Chapter 30

Fundamental Integration Formulas

IF $F(x)$ IS A FUNCTION whose derivative $F'(x) = f(x)$ on a certain interval of the x axis, then $F(x)$ is called an *antiderivative* or *indefinite integral* of $f(x)$. The indefinite integral of a given function is not unique; for example, x^2, $x^2 + 5$, and $x^2 - 4$ are all indefinite integrals of $f(x) = 2x$, since $\frac{d}{dx}(x^2) = \frac{d}{dx}(x^2 + 5) = \frac{d}{dx}(x^2 - 4) = 2x$. All indefinite integrals of $f(x) = 2x$ are then included in $F(x) = x^2 + C$, where C, called the *constant of integration*, is an arbitrary constant.

The symbol $\int f(x)\,dx$ is used to indicate the indefinite integral of $f(x)$. Thus we write $\int 2x\,dx = x^2 + C$. In the expression $\int f(x)\,dx$, the function $f(x)$ is called the *integrand*.

FUNDAMENTAL INTEGRATION FORMULAS. A number of the formulas below follow immediately from the standard differentiation formulas of earlier chapters, while others may be checked by differentiation. Formula 25, for example, may be checked by showing that

$$\frac{d}{dx}\left(\frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C\right) = \sqrt{a^2 - x^2}$$

Absolute value signs appear in certain of the formulas. For example, for formula 5 we write $\int \frac{dx}{x} = \ln |x| + C$ instead of

$$\int \frac{dx}{x} = \ln x + C \text{ for } x > 0 \qquad \text{and} \qquad \int \frac{dx}{x} = \ln(-x) + C \text{ for } x < 0$$

and for formula 10 we have $\int \tan x\,dx = \ln |\sec x| + C$ instead of

$$\int \tan x\,dx = \ln \sec x + C \qquad \text{for all } x \text{ such that } \sec x \geq 1$$

and

$$\int \tan x\,dx = \ln(-\sec x) + C \qquad \text{for all } x \text{ such that } \sec x \leq -1$$

1. $\int \frac{d}{dx}[f(x)]\,dx = f(x) + C$

2. $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

3. $\int af(x)\,dx = a\int f(x)\,dx$, $\quad a$ any constant

4. $\int x^m\,dx = \frac{x^{m+1}}{m+1} + C$, $\quad m \neq -1$

5. $\int \frac{dx}{x} = \ln |x| + C$

6. $\int a^x\,dx = \frac{a^x}{\ln a} + C$, $\quad a > 0, a \neq 1$

7. $\int e^x\,dx = e^x + C$

8. $\int \sin x\,dx = -\cos x + C$

9. $\int \cos x\, dx = \sin x + C$

10. $\int \tan x\, dx = \ln |\sec x| + C$

11. $\int \cot x\, dx = \ln |\sin x| + C$

12. $\int \sec x\, dx = \ln |\sec x + \tan x| + C$

13. $\int \csc x\, dx = \ln |\csc x - \cot x| + C$

14. $\int \sec^2 x\, dx = \tan x + C$

15. $\int \csc^2 x\, dx = -\cot x + C$

16. $\int \sec x \tan x\, dx = \sec x + C$

17. $\int \csc x \cot x\, dx = -\csc x + C$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$

19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$

21. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x - a}{x + a} \right| + C$

22. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C$

23. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln (x + \sqrt{x^2 + a^2}) + C$

24. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln |x + \sqrt{x^2 - a^2}| + C$

25. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \arcsin \frac{x}{a} + C$

26. $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \ln (x + \sqrt{x^2 + a^2}) + C$

27. $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{1}{2} a^2 \ln |x + \sqrt{x^2 - a^2}| + C$

THE METHOD OF SUBSTITUTION. To evaluate an antiderivative $\int f(x)\, dx$, it is often useful to replace x with a new variable u by means of a *substitution* $x = g(u)$, $dx = g'(u)\, du$. The equation

$$\int f(x)\, dx = \int f(g(u))g'(u)\, du \tag{30.1}$$

is valid. After finding the right side of (*30.1*), we replace u with $g^{-1}(x)$; that is, we obtain the result in terms of x. To verify (*30.1*), observe that, if $F(x) = \int f(x)\, dx$, then $\frac{d}{du} F(x) = \frac{d}{dx} F(x) \frac{dx}{du} = f(x)g'(u) = f(g(u))g'(u)$. Hence, $F(x) = \int f(g(u))g'(u)\, du$, which is (*30.1*).

EXAMPLE 1: To evaluate $\int (x + 3)^{11}\, dx$, replace $x + 3$ with u; that is, let $x = u - 3$. Then $dx = du$, and we obtain

$$\int (x + 3)^{11}\, dx = \int u^{11}\, du = \tfrac{1}{12}u^{12} + C = \tfrac{1}{12}(x + 3)^{12} + C$$

QUICK INTEGRATION BY INSPECTION. Two simple formulas enable us to find antiderivatives almost immediately. The first is

$$\int g'(x)[g(x)]^r\, dx = \frac{1}{r + 1} [g(x)]^{r+1} + C \quad r \neq -1 \tag{30.2}$$

This formula is justified by noting that $\frac{d}{dx}\left\{\frac{1}{r+1}[g(x)]^{r+1}\right\} = g'(x)[g(x)]^r$.

EXAMPLE 2: (*a*) $\int \frac{(\ln x)^2}{x}\,dx = \int \frac{1}{x}(\ln x)^2\,dx = \frac{1}{3}(\ln x)^3 + C$

(*b*) $\int x\sqrt{x^2+3}\,dx = \frac{1}{2}\int (2x)(x^2+3)^{1/2}\,dx = \frac{1}{2}\left[\frac{1}{3/2}(x^2+3)^{3/2}\right] + C = \frac{1}{3}[\sqrt{x^2+3}]^3 + C$

The second quick integration formula is

$$\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C \tag{30.3}$$

This formula is justified by noting that $\frac{d}{dx}(\ln|g(x)|) = \frac{g'(x)}{g(x)}$.

EXAMPLE 3: (*a*) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$

(*b*) $\int \frac{x^2}{x^3-5}\,dx = \frac{1}{3}\int \frac{3x^2}{x^3-5}\,dx = \frac{1}{3}\ln|x^3-5| + C$

Solved Problems

In Problems 1 to 8, evaluate the indefinite integral at the left.

1. $\int x^5\,dx = \frac{x^6}{6} + C$

2. $\int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C$

3. $\int \sqrt[3]{z}\,dz = \int z^{1/3}\,dz = \frac{z^{4/3}}{4/3} + C = \frac{3}{4}z^{4/3} + C$

4. $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}\,dx = \frac{x^{1/3}}{1/3} + C = 3x^{1/3} + C$

5. $\int (2x^2 - 5x + 3)\,dx = 2\int x^2\,dx - 5\int x\,dx + 3\int dx = \frac{2x^3}{3} - \frac{5x^2}{2} + 3x + C$

6. $\int (1-x)\sqrt{x}\,dx = \int (x^{1/2} - x^{3/2})\,dx = \int x^{1/2}\,dx - \int x^{3/2}\,dx = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

7. $\int (3s+4)^2\,ds = \int (9s^2 + 24s + 16)\,ds = 9(\frac{1}{3}s^3) + 24(\frac{1}{2}s^2) + 16s + C = 3s^3 + 12s^2 + 16s + C$

8. $\int \frac{x^3 + 5x^2 - 4}{x^2}\,dx = \int (x + 5 - 4x^{-2})\,dx = \frac{1}{2}x^2 + 5x - \frac{4x^{-1}}{-1} + C = \frac{1}{2}x^2 + 5x + \frac{4}{x} + C$

9. Evaluate (*a*) $\int (x^3+2)^2(3x^2)\,dx$, (*b*) $\int (x^3+2)^{1/2}x^2\,dx$, (*c*) $\int \frac{8x^2\,dx}{(x^3+2)^3}$, and (*d*) $\int \frac{x^2\,dx}{\sqrt[4]{(x^3+2)}}$ by means of (*30.2*).

(a) $\int (x^3+2)^2(3x^2)\,dx = \frac{1}{3}(x^3+2)^3 + C$

(b) $\int (x^3+2)^{1/2}x^2\,dx = \frac{1}{3}\int (x^3+2)^{1/2}(3x^2)\,dx = \frac{1}{3}\frac{2}{3}(x^3+2)^{3/2} + C = \frac{2}{9}(x^3+2)^{3/2} + C$

(c) $\int \frac{8x^2}{(x^3+2)^3}\,dx = \frac{8}{3}\int (x^3+2)^{-3}(3x^2)\,dx = \frac{8}{3}\left(-\frac{1}{2}\right)(x^3+2)^{-2} + C = -\frac{4}{3}\,\frac{1}{(x^3+2)^2} + C$

(d) $\int \frac{x^2}{\sqrt[4]{x^3+2}}\,dx = \frac{1}{3}\int (x^3+2)^{-1/4}(3x^2)\,dx = \frac{1}{3}\,\frac{4}{3}\,(x^3+2)^{3/4} + C = \frac{4}{9}\,(x^3+2)^{3/4} + C$

All four integrals can also be evaluated by making the substitution $u = x^3+2$, $du = 3x^2\,dx$.

10. Evaluate $\int 3x\sqrt{1-2x^2}\,dx$.

Formula (*30.2*) yilelds

$$\int 3x\sqrt{1-2x^2}\,dx = 3(-\tfrac{1}{4})\int (1-2x^2)^{1/2}(-4x)\,dx = -\tfrac{3}{4}\tfrac{2}{3}(1-2x^2)^{3/2} + C$$
$$= -\tfrac{1}{2}(1-2x^2)^{3/2} + C$$

We could also use the substitution $u = 1-2x^2$, $du = -4x\,dx$.

11. Evaluate $\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}}$.

Formula (*30.2*) yields

$$\int \frac{(x+3)\,dx}{(x^2+6x)^{1/3}} = \frac{1}{2}\int (x^2+6x)^{-1/3}(2x+6)\,dx = \frac{1}{2}\,\frac{3}{2}\,(x^2+6x)^{2/3} + C$$
$$= \frac{3}{4}\,(x^2+6x)^{2/3} + C$$

We could also use the substitution $u = x^2+6x$, $du = (2x+6)\,dx$.

In Problems 12 to 15, evaluate the indefinite integral on the left.

12. $\int \sqrt[3]{1-x^2}\,x\,dx = -\frac{1}{2}\int (1-x^2)^{1/3}(-2x\,dx) = -\frac{1}{2}\frac{3}{4}(1-x^2)^{4/3} + C = -\frac{3}{8}(1-x^2)^{4/3} + C$

13. $\int \sqrt{x^2-2x^4}\,dx = \int (1-2x^2)^{1/2}x\,dx = -\frac{1}{4}\int (1-2x^2)^{1/2}(-4x\,dx) - \frac{1}{4}\frac{2}{3}(1-2x^2)^{3/2} + C$
$= -\frac{1}{6}(1-2x^2)^{3/2} + C$

14. $\int \frac{(1+x)^2}{\sqrt{x}}\,dx = \int \frac{1+2x+x^2}{x^{1/2}}\,dx = \int (x^{-1/2} + 2x^{1/2} + x^{3/2})\,dx = 2x^{1/2} + \frac{4}{3}\,x^{3/2} + \frac{2}{5}\,x^{5/2} + C$

15. $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \left[1 - \frac{1}{(x+1)^2}\right]dx = x + \frac{1}{x+1} + C' = \frac{x^2}{x+1} + 1 + C' = \frac{x^2}{x+1} + C$

FORMULAS 5 TO 7

16. Evaluate $\int dx/x$.

Formula 5 gives $\int \frac{dx}{x} = \ln|x| + C$.

17. Evaluate $\int \frac{dx}{x+2}$, using *(30.3)*.

$\int \frac{dx}{x+2} = \ln|x+2| + C$. We also could use formula 5 and the substitution $u = x + 2$, $du = dx$.

18. Evaluate $\int \frac{dx}{2x-3}$, using *(30.3)*.

$\int \frac{dx}{2x-3} = \frac{1}{2}\int \frac{2\,dx}{2x-3} = \frac{1}{2}\ln|2x-3| + C$. Another method is to make the substitution $u = 2x - 3$, $du = 2\,dx$.

In Problems 19 to 27, evaluate the integral at the left.

19. $\int \frac{x\,dx}{x^2-1} = \frac{1}{2}\int \frac{2x\,dx}{x^2-1} = \frac{1}{2}\ln|x^2-1| + C = \frac{1}{2}\ln|x^2-1| + \ln c = \ln(c\sqrt{|x^2-1|})$

20. $\int \frac{x^2\,dx}{1-2x^3} = -\frac{1}{6}\int \frac{-6x^2\,dx}{1-2x^3} = -\frac{1}{6}\ln|1-2x^3| + C = \ln\frac{c}{\sqrt[6]{|1-2x^3|}}$

21. $\int \frac{x+2}{x+1}\,dx = \int \left(1 + \frac{1}{x+1}\right)dx = x + \ln|x+1| + C$

22. $\int e^{-x}\,dx = -\int e^{-x}(-dx) = -e^{-x} + C$

23. $\int a^{2x}\,dx = \frac{1}{2}\int a^{2x}(2\,dx) = \frac{1}{2}\frac{a^{2x}}{\ln a} + C$

24. $\int e^{3x}\,dx = \frac{1}{3}\int e^{3x}(3\,dx) = \frac{e^{3x}}{3} + C$

25. $\int \frac{e^{1/x}\,dx}{x^2} = -\int e^{1/x}\left(-\frac{dx}{x^2}\right) = -e^{1/x} + C$

26. $\int (e^x+1)^3 e^x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{(e^x+1)^4}{4} + C$, where $u = e^x + 1$ and $du = e^x\,dx$, or

$$\int (e^x+1)^3 e^x\,dx = \int (e^x+1)^3\,d(e^x+1) = \frac{(e^x+1)^4}{4} + C$$

27. $\int \frac{dx}{e^x+1} = \int \frac{e^{-x}\,dx}{1+e^{-x}} = \int \frac{-e^{-x}\,dx}{1+e^{-x}} = -\ln(1+e^{-x}) + C = \ln\frac{e^x}{1+e^x} + C$

$$= x - \ln(1+e^x) + C$$

The absolute-value sign is not needed here because $1 + e^{-x} > 0$ for all values of x.

FORMULAS 8 TO 17

In Problems 28 to 47, evaluate the integral at the left.

28. $\int \sin\frac{1}{2}x\,dx = 2\int (\sin\frac{1}{2}x)(\frac{1}{2}\,dx) = -2\cos\frac{1}{2}x + C$

29. $\int \cos 3x\,dx = \frac{1}{3}\int (\cos 3x)(3\,dx) = \frac{1}{3}\sin 3x + C$

30. $\int \sin^2 x\cos x\,dx = \int \sin^2 x(\cos x\,dx) = \dfrac{\sin^3 x}{3} + C$

31. $\int \tan x\,dx = \int \dfrac{\sin x}{\cos x}\,dx = -\int \dfrac{-\sin x\,dx}{\cos x} = -\ln|\cos x| + C = \ln|\sec x| + C$

32. $\int \tan 2x\,dx = \frac{1}{2}\int (\tan 2x)(2\,dx) = \frac{1}{2}\ln|\sec 2x| + C$

33. $\int x\cot x^2\,dx = \frac{1}{2}\int (\cot x^2)(2x\,dx) = \frac{1}{2}\ln|\sin x^2| + C$

34. $\int \sec x\,dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \dfrac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}\,dx = \ln|\sec x + \tan x| + C$

35. $\int \sec\sqrt{x}\,\dfrac{dx}{\sqrt{x}} = 2\int (\sec x^{1/2})(\frac{1}{2}x^{-1/2}\,dx) = 2\ln|\sec\sqrt{x} + \tan\sqrt{x}| + C$

36. $\int \sec^2 2ax\,dx = \dfrac{1}{2a}\int (\sec^2 2ax)(2a\,dx) = \dfrac{\tan 2ax}{2a} + C$

37. $\int \dfrac{\sin x + \cos x}{\cos x}\,dx = \int (\tan x + 1)\,dx = \ln|\sec x| + x + C$

38. $\int \dfrac{\sin y\,dy}{\cos^2 y} = \int \tan y\sec y\,dy = \sec y + C$

39. $\int (1 + \tan x)^2\,dx = \int (1 + 2\tan x + \tan^2 x)\,dx = \int (\sec^2 x + 2\tan x)\,dx$

$= \tan x + 2\ln|\sec x| + C$

40. $\int e^x\cos e^x\,dx = \int (\cos e^x)(e^x\,dx) = \sin e^x + C$

41. $\int e^{3\cos 2x}\sin 2x\,dx = -\dfrac{1}{6}\int e^{3\cos 2x}(-6\sin 2x\,dx) = -\dfrac{e^{3\cos 2x}}{6} + C$

42. $\int \dfrac{dx}{1 + \cos x} = \int \dfrac{1 - \cos x}{1 - \cos^2 x}\,dx = \int \dfrac{1 - \cos x}{\sin^2 x}\,dx = \int (\csc^2 x - \cot x\csc x)\,dx$

$= -\cot x + \csc x + C$

43. $\int (\tan 2x + \sec 2x)^2\,dx = \int (\tan^2 2x + 2\tan 2x\sec 2x + \sec^2 2x)\,dx$

$= \int (2\sec^2 2x + 2\tan 2x\sec 2x - 1)\,dx = \tan 2x + \sec 2x - x + C$

44. $\int \csc u\,du = \int \dfrac{du}{\sin u} = \int \dfrac{du}{2\sin\frac{1}{2}u\cos\frac{1}{2}u} = \int \dfrac{(\sec^2\frac{1}{2}u)(\frac{1}{2}\,du)}{\tan\frac{1}{2}u} = \ln|\tan\tfrac{1}{2}u| + C$

45. $\int (\sec 4x - 1)^2\,dx = \int (\sec^2 4x - 2\sec 4x + 1)\,dx = \frac{1}{4}\tan 4x - \frac{1}{2}\ln|\sec 4x + \tan 4x| + x + C$

46. $\int \frac{\sec x \tan x\,dx}{a + b\sec x} = \frac{1}{b}\int \frac{(\sec x \tan x)(b\,dx)}{a + b\sec x} = \frac{1}{b}\ln|a + b\sec x| + C$

47. $\int \frac{dx}{\csc 2x - \cot 2x} = \int \frac{\sin 2x\,dx}{1 - \cos 2x} = \frac{1}{2}\int \frac{(\sin 2x)(2\,dx)}{1 - \cos 2x} = \frac{1}{2}\ln(1 - \cos 2x) + C'$

$$= \frac{1}{2}\ln(2\sin^2 x) + C' = \frac{1}{2}(\ln 2 + 2\ln|\sin x|) + C' = \ln|\sin x| + C$$

FORMULAS 18 TO 20

In Problems 48 to 72, evaluate the integral at the left.

48. $\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C$

49. $\int \frac{dx}{1 + x^2} = \arctan x + C$

50. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \operatorname{arcsec} x + C$

51. $\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$

52. $\int \frac{dx}{9 + x^2} = \frac{1}{3}\arctan \frac{x}{3} + C$

53. $\int \frac{dx}{\sqrt{25 - 16x^2}} = \frac{1}{4}\int \frac{4\,dx}{\sqrt{5^2 - (4x)^2}} = \frac{1}{4}\arcsin \frac{4x}{5} + C$

54. $\int \frac{dx}{4x^2 + 9} = \frac{1}{2}\int \frac{2\,dx}{(2x)^2 + 3^2} = \frac{1}{6}\arctan \frac{2x}{3} + C$

55. $\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2 - 3^2}} = \frac{1}{3}\operatorname{arcsec} \frac{2x}{3} + C$

56. $\int \frac{x^2\,dx}{\sqrt{1 - x^6}} = \frac{1}{3}\int \frac{3x^2\,dx}{\sqrt{1 - (x^3)^2}} = \frac{1}{3}\arcsin x^3 + C$

57. $\int \frac{x\,dx}{x^4 + 3} = \frac{1}{2}\int \frac{2x\,dx}{(x^2)^2 + 3} = \frac{1}{2}\frac{1}{\sqrt{3}}\arctan \frac{x^2}{\sqrt{3}} + C = \frac{\sqrt{3}}{6}\arctan \frac{x^2\sqrt{3}}{3} + C$

58. $\int \frac{dx}{x\sqrt{x^4 - 1}} = \frac{1}{2}\int \frac{2x\,dx}{x^2\sqrt{(x^2)^2 - 1}} = \frac{1}{2}\operatorname{arcsec} x^2 + C = \frac{1}{2}\arccos \frac{1}{x^2} + C$

59. $\int \frac{dx}{\sqrt{4 - (x + 2)^2}} = \arcsin \frac{x + 2}{2} + C$

60. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1} = \arctan e^x + C$

61. $\int \frac{3x^3 - 4x^2 + 3x}{x^2 + 1}\,dx = \int \left(3x - 4 + \frac{4}{x^2 + 1}\right) dx = \frac{3x^2}{2} - 4x + 4\arctan x + C$

62. $$\int \frac{\sec x \tan x\, dx}{9 + 4\sec^2 x} = \frac{1}{2}\int \frac{2\sec x \tan x\, dx}{3^2 + (2\sec x)^2} = \frac{1}{6}\arctan\frac{2\sec x}{3} + C$$

63. $$\int \frac{(x+3)\, dx}{\sqrt{1-x^2}} = \int \frac{x\, dx}{\sqrt{1-x^2}} + 3\int \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + 3\arcsin x + C$$

64. $$\int \frac{(2x-7)\, dx}{x^2+9} = \int \frac{2x\, dx}{x^2+9} - 7\int \frac{dx}{x^2+9} = \ln(x^2+9) - \frac{7}{3}\arctan\frac{x}{3} + C$$

65. $$\int \frac{dy}{y^2+10y+30} = \int \frac{dy}{(y^2+10y+25)+5} = \int \frac{dy}{(y+5)^2+5} = \frac{\sqrt{5}}{5}\arctan\frac{(y+5)\sqrt{5}}{5} + C$$

66. $$\int \frac{dx}{\sqrt{20+8x-x^2}} = \int \frac{dx}{\sqrt{36-(x^2-8x+16)}} = \int \frac{dx}{\sqrt{36-(x-4)^2}} = \arcsin\frac{x-4}{6} + C$$

67. $$\int \frac{dx}{2x^2+2x+5} = \int \frac{2\, dx}{4x^2+4x+10} = \int \frac{2\, dx}{(2x+1)^2+9} = \frac{1}{3}\arctan\frac{2x+1}{3} + C$$

68. $$\int \frac{x+1}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{2x+2}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{(2x-4)+6}{x^2-4x+8}\, dx = \frac{1}{2}\int \frac{(2x-4)\, dx}{x^2-4x+8} + 3\int \frac{dx}{x^2-4x+8}$$
$$= \frac{1}{2}\int \frac{(2x-4)\, dx}{x^2-4x+8} + 3\int \frac{dx}{(x-2)^2+4} = \frac{1}{2}\ln(x^2-4x+8) + \frac{3}{2}\arctan\frac{x-2}{2} + C$$

The absolute-value sign is not needed here because $x^2 - 4x + 8 > 0$ for all values of x.

69. $$\int \frac{dx}{\sqrt{28-12x-x^2}} = \int \frac{dx}{\sqrt{64-(x^2+12x+36)}} = \int \frac{dx}{\sqrt{64-(x+6)^2}} = \arcsin\frac{x+6}{8} + C$$

70. $$\int \frac{x+3}{\sqrt{5-4x-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x-6}{\sqrt{5-4x-x^2}}\, dx = -\frac{1}{2}\int \frac{(-2x-4)-2}{\sqrt{5-4x-x^2}}\, dx$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\, dx + \int \frac{dx}{\sqrt{5-4x-x^2}}$$
$$= -\frac{1}{2}\int \frac{-2x-4}{\sqrt{5-4x-x^2}}\, dx + \int \frac{dx}{\sqrt{9-(x+2)^2}}$$
$$= -\sqrt{5-4x-x^2} + \arcsin\frac{x+2}{3} + C$$

71. $$\int \frac{2x+3}{9x^2-12x+8}\, dx = \frac{1}{9}\int \frac{18x+27}{9x^2-12x+8}\, dx = \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}\, dx$$
$$= \frac{1}{9}\int \frac{18x-12}{9x^2-12x+8}\, dx + \frac{13}{3}\int \frac{dx}{(3x-2)^2+4}$$
$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{18}\arctan\frac{3x-2}{2} + C$$

72. $$\int \frac{x+2}{\sqrt{4x-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\, dx = -\frac{1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}\, dx$$
$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\, dx + 4\int \frac{dx}{\sqrt{4-(x-2)^2}} = -\sqrt{4x-x^2} + 4\arcsin\frac{x-2}{2} + C$$

FORMULAS 21 TO 24

In Problems 73 to 89, evaluate the integral at the left.

73. $\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

74. $\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

75. $\int \frac{dx}{x^2-4} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

76. $\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$

77. $\int \frac{dx}{\sqrt{x^2+1}} = \ln(x+\sqrt{x^2+1}) + C$

78. $\int \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}| + C$

79. $\int \frac{dx}{\sqrt{4x^2+9}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln(2x+\sqrt{4x^2+9}) + C$

80. $\int \frac{dz}{\sqrt{9z^2-25}} = \frac{1}{3}\int \frac{3\,dz}{\sqrt{9z^2-25}} = \frac{1}{3}\ln|3z+\sqrt{9z^2-25}| + C$

81. $\int \frac{dx}{9x^2-16} = \frac{1}{3}\int \frac{3\,dx}{(3x)^2-16} = \frac{1}{24}\ln\left|\frac{3x-4}{3x+4}\right| + C$

82. $\int \frac{dy}{25-16y^2} = \frac{1}{4}\int \frac{4\,dy}{25-(4y)^2} = \frac{1}{40}\ln\left|\frac{5+4y}{5-4y}\right| + C$

83. $\int \frac{dx}{x^2+6x+8} = \int \frac{dx}{(x+3)^2-1} = \frac{1}{2}\ln\left|\frac{(x+3)-1}{(x+3)+1}\right| + C = \frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C$

84. $\int \frac{dx}{4x-x^2} = \int \frac{dx}{4-(x-2)^2} = \frac{1}{4}\ln\left|\frac{2+(x-2)}{2-(x-2)}\right| + C = \frac{1}{4}\ln\left|\frac{x}{4-x}\right| + C$

85. $\int \frac{ds}{\sqrt{4s+s^2}} = \int \frac{ds}{\sqrt{(s+2)^2-4}} = \ln|s+2+\sqrt{4s+s^2}| + C$

86. $\int \frac{x+2}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+9}}\,dx = \frac{1}{2}\int \frac{2x\,dx}{\sqrt{x^2+9}} + 2\int \frac{dx}{\sqrt{x^2+9}}$

$= \sqrt{x^2+9} + 2\ln(x+\sqrt{x^2+9}) + C$

87. $\int \frac{2x-3}{4x^2-11}\,dx = \frac{1}{4}\int \frac{8x-12}{4x^2-11}\,dx = \frac{1}{4}\int \frac{8x\,dx}{4x^2-11} - \frac{3}{2}\int \frac{2\,dx}{4x^2-11}$

$= \frac{1}{4}\ln|4x^2-11| - \frac{3\sqrt{11}}{44}\ln\left|\frac{2x-\sqrt{11}}{2x+\sqrt{11}}\right| + C$

88. $\int \frac{x+2}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+2x-3}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-3}}\,dx + \int \frac{dx}{\sqrt{(x+1)^2-4}}$

$= \sqrt{x^2+2x-3} + \ln|x+1+\sqrt{x^2+2x-3}| + C$

89. $\displaystyle\int \frac{2-x}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x-16}{4x^2+4x-3}\,dx = -\frac{1}{8}\int \frac{8x+4}{4x^2+4x-3}\,dx + \frac{5}{2}\int \frac{dx}{(2x+1)^2-4}$

$$= -\frac{1}{8}\ln|4x^2+4x-3| + \frac{5}{16}\ln\left|\frac{2x-1}{2x+3}\right| + C$$

FORMULAS 25 TO 27

In Problems 90 to 95, evaluate the integral at the left.

90. $\displaystyle\int \sqrt{25-x^2}\,dx = \frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\arcsin\frac{x}{5} + C$

91. $\displaystyle\int \sqrt{3-4x^2}\,dx = \frac{1}{2}\int (\sqrt{3-4x^2})(2\,dx) = \frac{1}{2}\left(\frac{2x}{2}\sqrt{3-4x^2} + \frac{3}{2}\arcsin\frac{2x}{\sqrt{3}}\right) + C$

$$= \frac{1}{2}x\sqrt{3-4x^2} + \frac{3}{4}\arcsin\frac{2x\sqrt{3}}{3} + C$$

92. $\displaystyle\int \sqrt{x^2-36}\,dx = \tfrac{1}{2}x\sqrt{x^2-36} - 18\ln|x+\sqrt{x^2-36}| + C$

93. $\displaystyle\int \sqrt{3x^2+5}\,dx = \frac{1}{\sqrt{3}}\int \sqrt{3x^2+5}\sqrt{3}\,dx = \frac{1}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}x\sqrt{3x^2+5} + \frac{5}{2}\ln(\sqrt{3}x+\sqrt{3x^2+5})\right] + C$

$$= \frac{1}{2}x\sqrt{3x^2+5} + \frac{5\sqrt{3}}{6}\ln(\sqrt{3}x+\sqrt{3x^2+5}) + C$$

94. $\displaystyle\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{4-(x+1)^2}\,dx = \frac{x+1}{2}\sqrt{3-2x-x^2} + 2\arcsin\frac{x+1}{2} + C$

95. $\displaystyle\int \sqrt{4x^2-4x+5}\,dx = \frac{1}{2}\int (\sqrt{(2x-1)^2+4})(2\,dx)$

$$= \frac{1}{2}\left[\frac{2x-1}{2}\sqrt{4x^2-4x+5} + 2\ln(2x-1+\sqrt{4x^2-4x+5})\right] + C$$

$$= \frac{2x-1}{4}\sqrt{4x^2-4x+5} + \ln(2x-1+\sqrt{4x^2-4x+5}) + C$$

Supplementary Problems

In Problems 96 to 200, evaluate the integral at the left.

96. $\displaystyle\int (4x^3+3x^2+2x+5)\,dx = x^4+x^3+x^2+5x+C$

97. $\displaystyle\int (3-2x-x^4)\,dx = 3x-x^2-\tfrac{1}{5}x^5+C$

98. $\displaystyle\int (2-3x+x^3)\,dx = 2x-\tfrac{3}{2}x^2+\tfrac{1}{4}x^4+C$

99. $\displaystyle\int (x^2-1)^2\,dx = x^5/5-2x^3/3+x+C$

100. $\int (\sqrt{x} - \frac{1}{2}x + 2/\sqrt{x})\,dx = \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + C$

101. $\int (a+x)^3\,dx = \frac{1}{4}(a+x)^4 + C$

102. $\int (x-2)^{3/2}\,dx = \frac{2}{5}(x-2)^{5/2} + C$

103. $\int \frac{dx}{x^3} = -\frac{1}{2x^2} + C$

104. $\int \frac{dx}{(x-1)^3} = -\frac{1}{2(x-1)^2} + C$

105. $\int \frac{dx}{\sqrt{x+3}} = 2\sqrt{x+3} + C$

106. $\int \sqrt{3x-1}\,dx = \frac{2}{9}(3x-1)^{3/2} + C$

107. $\int \sqrt{2-3x}\,dx = -\frac{2}{9}(2-3x)^{3/2} + C$

108. $\int (2x^2+3)^{1/3}x\,dx = \frac{3}{16}(2x^2+3)^{4/3} + C$

109. $\int (x-1)^2 x\,dx = \frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + C$

110. $\int (x^2-1)x\,dx = \frac{1}{4}(x^2-1)^2 + C$

111. $\int \sqrt{1+y^4}\,y^3\,dy = \frac{1}{6}(1+y^4)^{3/2} + C$

112. $\int (x^3+3)x^2\,dx = \frac{1}{6}(x^3+3)^2 + C$

113. $\int (4-x^2)^2x^2\,dx = \frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{1}{7}x^7 + C$

114. $\int \frac{dy}{(2-y)^3} = \frac{1}{2(2-y)^2} + C$

115. $\int \frac{x\,dx}{(x^2+4)^3} = -\frac{1}{4(x^2+4)^2} + C$

116. $\int (1-x^3)^2\,dx = x - \frac{1}{2}x^4 + \frac{1}{7}x^7 + C$

117. $\int (1-x^3)^2 x\,dx = \frac{1}{2}x^2 - \frac{2}{5}x^5 + \frac{1}{8}x^8 + C$

118. $\int (1-x^3)^2x^2\,dx = -\frac{1}{9}(1-x^3)^3 + C$

119. $\int (x^2-x)^4(2x-1)\,dx = \frac{1}{5}(x^2-x)^5 + C$

120. $\int \frac{3t\,dt}{\sqrt[3]{t^2+3}} = \frac{9}{4}(t^2+3)^{2/3} + C$

121. $\int \frac{(x+1)\,dx}{\sqrt{x^2+2x-4}} = \sqrt{x^2+2x-4} + C$

122. $\int \frac{dx}{(a+bx)^{1/3}} = \frac{3}{2b}(a+bx)^{2/3} + C$

123. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx = \frac{2}{3}(1+\sqrt{x})^3 + C$

124. $\int \sqrt{x}\,(3-5x)\,dx = 2x^{3/2}(1-x) + C$

125. $\int \frac{(x+1)(x-2)}{\sqrt{x}}\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} - 4x^{1/2} + C$

126. $\int \frac{dx}{x-1} = \ln|x-1| + C$

127. $\int \frac{dx}{3x+1} = \frac{1}{3}\ln|3x+1| + C$

128. $\int \frac{3x\,dx}{x^2+2} = \frac{3}{2}\ln(x^2+2) + C$

129. $\int \frac{x^2\,dx}{1-x^3} = -\frac{1}{3}\ln|1-x^3| + C$

130. $\int \frac{x-1}{x+1}\,dx = x - 2\ln|x+1| + C$

131. $\int \frac{x^2+2x+2}{x+2}\,dx = \frac{1}{2}x^2 + 2\ln|x+2| + C$

132. $\int \frac{x+1}{x^2+2x+2}\,dx = \frac{1}{2}\ln(x^2+2x+2) + C$

133. $\int \left(\frac{dx}{2x-1} - \frac{dx}{2x+1}\right) = \ln\sqrt{\left|\frac{2x-1}{2x+1}\right|} + C$

134. $\int a^{4x}\,dx = \frac{1}{4}\frac{a^{4x}}{\ln a} + C$

135. $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$

136. $\int \frac{e^{1/x^2}}{x^3}\,dx = -\frac{1}{2}e^{1/x^2} + C$

137. $\int e^{-x^2+2}x\,dx = -\frac{1}{2}e^{-x^2+2} + C$

138. $\int x^2e^{x^3}\,dx = \frac{1}{3}e^{x^3} + C$

139. $\int (e^x+1)^2\,dx = \frac{1}{2}e^{2x} + 2e^x + x + C$

140. $\int (e^x - x^e)\,dx = e^x - \frac{x^{e+1}}{e+1} + C$

141. $\int (e^x+1)^2 e^x\,dx = \frac{1}{3}(e^x+1)^3 + C$

142. $\int \frac{e^{2x}}{e^{2x}+3}\,dx = \frac{1}{2}\ln(e^{2x}+3) + C$

143. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2e^{2x}} + C$

144. $\int \frac{e^x-1}{e^x+1}\,dx = \ln(e^x+1)^2 - x + C$

145. $\int \frac{e^{2x}-1}{e^{2x}+3}\,dx = \ln(e^{2x}+3)^{2/3} - \frac{1}{3}x + C$

146. $\int \frac{dx}{\sqrt{x}(1-\sqrt{x})} = \ln\frac{C}{(1-\sqrt{x})^2}$, $C>0$

147. $\int \frac{dx}{x+x^{1/3}} = \frac{3}{2}\ln C(x^{2/3}+1)$, $C>0$

148. $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

149. $\int \cos\frac{1}{2}x\,dx = 2\sin\frac{1}{2}x + C$

150. $\int \sec 3x\tan 3x\,dx = \frac{1}{3}\sec 3x + C$

151. $\int \csc^2 2x\,dx = -\frac{1}{2}\cot 2x + C$

152. $\int x\sec^2 x^2\,dx = \frac{1}{2}\tan x^2 + C$

153. $\int \tan^2 x\,dx = \tan x - x + C$

154. $\int \tan\frac{1}{2}x\,dx = 2\ln|\sec\frac{1}{2}x| + C$

155. $\int \csc 3x\,dx = \frac{1}{3}\ln|\csc 3x - \cot 3x| + C$

156. $\int b\sec ax\tan ax\,dx = \frac{b}{a}\sec ax + C$

157. $\int (\cos x - \sin x)^2\,dx = x + \frac{1}{2}\cos 2x + C$

158. $\int \sin ax\cos ax\,dx = \frac{1}{2a}\sin^2 ax + C$

$= -\frac{1}{2a}\cos^2 ax + C' = -\frac{1}{4a}\cos 2ax + C''$

159. $\int \sin^3 x\cos x\,dx = \frac{1}{4}\sin^4 x + C$

160. $\int \cos^4 x\sin x\,dx = -\frac{1}{5}\cos^5 x + C$

161. $\int \tan^5 x\sec^2 x\,dx = \frac{1}{6}\tan^6 x + C$

162. $\int \cot^4 3x\csc^2 3x\,dx = -\frac{1}{15}\cot^5 3x + C$

163. $\int \frac{dx}{1-\sin\frac{1}{2}x} = 2(\tan\frac{1}{2}x + \sec\frac{1}{2}x) + C$

164. $\int \frac{dx}{1+\cos 3x} = \frac{1-\cos 3x}{3\sin 3x} + C$

165. $\int \frac{dx}{1+\sec ax} = x + \frac{1}{a}(\cot ax - \csc ax) + C$

166. $\int \sec^2\frac{x}{a}\tan\frac{x}{a}\,dx = \frac{1}{2}a\tan^2\frac{x}{a} + C$

167. $\int \frac{\sec^2 3x}{\tan 3x}\,dx = \frac{1}{3}\ln|\tan 3x| + C$

168. $\int \frac{\sec^5 x}{\csc x}\,dx = \frac{1}{4}\sec^4 x + C$

169. $\int e^{\tan 2x}\sec^2 2x\,dx = \frac{1}{2}e^{\tan 2x} + C$

170. $\int e^{2\sin 3x}\cos 3x\,dx = \frac{1}{6}e^{2\sin 3x} + C$

171. $\int \frac{dx}{\sqrt{5-x^2}} = \arcsin\frac{x\sqrt{5}}{5} + C$

172. $\int \frac{dx}{5+x^2} = \frac{\sqrt{5}}{5}\arctan\frac{x\sqrt{5}}{5} + C$

173. $\int \frac{dx}{x\sqrt{x^2-5}} = \frac{\sqrt{5}}{5}\operatorname{arcsec}\frac{x\sqrt{5}}{5} + C$

174. $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}} = \arcsin e^x + C$

175. $\int \frac{e^{2x}\,dx}{1+e^{4x}} = \frac{1}{2}\arctan e^{2x} + C$

176. $\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\frac{3x}{2} + C$

177. $\int \frac{dx}{9x^2+4} = \frac{1}{6}\arctan\frac{3x}{2} + C$

178. $\int \frac{\sin 8x}{9+\sin^4 4x}\,dx = \frac{1}{12}\arctan\frac{\sin^2 4x}{3} + C$

179. $\int \frac{\sec^2 x\,dx}{\sqrt{1-4\tan^2 x}} = \frac{1}{2}\arcsin(2\tan x) + C$

180. $\int \frac{dx}{x\sqrt{4-9\ln^2 x}} = \frac{1}{3}\arcsin\ln x^{3/2} + C$

181. $\int \frac{2x^4 - x^2}{2x^2 + 1}\,dx = \frac{1}{3}x^3 - x + \frac{\sqrt{2}}{2}\arctan x\sqrt{2} + C$

182. $\int \frac{\cos 2x\,dx}{\sin^2 2x + 8} = \frac{\sqrt{2}}{8}\arctan\frac{\sin 2x}{2\sqrt{2}} + C$

183. $\int \frac{(2x-3)\,dx}{x^2+6x+13} = \int \frac{(2x+6)\,dx}{x^2+6x+13} - 9\int \frac{dx}{x^2+6x+13} = \ln(x^2+6x+13) - \frac{9}{2}\arctan\frac{x+3}{2} + C$

184. $\int \frac{(x-1)\,dx}{3x^2-4x+3} = \frac{1}{6}\int \frac{(6x-4)\,dx}{3x^2-4x+3} - \int \frac{dx}{9x^2-12x+9} = \frac{1}{6}\ln(3x^2-4x+3) - \frac{\sqrt{5}}{15}\arctan\frac{3x-2}{\sqrt{5}} + C$

185. $\int \frac{x\,dx}{\sqrt{27+6x-x^2}} = -\sqrt{27+6x-x^2} + 3\arcsin\frac{x-3}{6} + C$

186. $\int \frac{(5-4x)\,dx}{\sqrt{12x-4x^2-8}} = \sqrt{12x-4x^2-8} - \frac{1}{2}\arcsin(2x-3) + C$

187. $\int \frac{dx}{x^2-4} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

188. $\int \frac{dx}{4x^2-9} = \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$

189. $\int \frac{dx}{9-x^2} = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

190. $\int \frac{dx}{25-9x^2} = \frac{1}{30}\ln\left|\frac{3x+5}{3x-5}\right| + C$

191. $\int \frac{dx}{\sqrt{x^2+4}} = \ln(x+\sqrt{x^2+4}) + C$

192. $\int \frac{dx}{\sqrt{4x^2-25}} = \frac{1}{2}\ln|2x+\sqrt{4x^2-25}| + C$

193. $\int \sqrt{16-9x^2}\,dx = \frac{1}{2}x\sqrt{16-9x^2} + \frac{8}{3}\arcsin\frac{3x}{4} + C$

194. $\int \sqrt{x^2-16}\,dx = \frac{1}{2}x\sqrt{x^2-16} - 8\ln|x+\sqrt{x^2-16}| + C$

195. $\int \sqrt{4x^2+9}\,dx = \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\ln(2x+\sqrt{4x^2+9}) + C$

196. $\int \sqrt{x^2-2x-3}\,dx = \frac{1}{2}(x-1)\sqrt{x^2-2x-3} - 2\ln|x-1+\sqrt{x^2-2x-3}| + C$

197. $\int \sqrt{12+4x-x^2}\,dx = \frac{1}{2}(x-2)\sqrt{12+4x-x^2} + 8\arcsin\frac{1}{4}(x-2) + C$

198. $\int \sqrt{x^2+4x}\,dx = \frac{1}{2}(x+2)\sqrt{x^2+4x} - 2\ln|x+2+\sqrt{x^2+4x}| + C$

199. $\int \sqrt{x^2-8x}\,dx = \frac{1}{2}(x-4)\sqrt{x^2-8x} - 8\ln|x-4+\sqrt{x^2-8x}| + C$

200. $\int \sqrt{6x-x^2}\,dx = \frac{1}{2}(x-3)\sqrt{6x-x^2} + \frac{9}{2}\arcsin\frac{x-3}{3} + C$

Chapter 31

Integration by Parts

INTEGRATION BY PARTS. When u and v are differentiable functions of x,

$$d(uv) = u\,dv + v\,du$$

or

$$u\,dv = d(uv) - v\,du$$

and

$$\int u\,dv = uv - \int v\,du \tag{31.1}$$

When (*31.1*) is to be used in a required integration, the given integral must be separated into two parts, one part being u and the other part, together with dx, being dv. (For this reason, integration by use of (*31.1*) is called *integration by parts*.) Two general rules can be stated:

1. The part selected as dv must be readily integrable.
2. $\int v\,du$ must not be more complex than $\int u\,dv$.

EXAMPLE 1: Find $\int x^3 e^{x^2}\,dx$.

Take $u = x^2$ and $dv = e^{x^2} x\,dx$; then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$. Now by (*31.1*),

$$\int x^3 e^{x^2}\,dx = \tfrac{1}{2}x^2 e^{x^2} - \int x\,e^{x^2}\,dx = \tfrac{1}{2}x^2\,e^{x^2} - \tfrac{1}{2}e^{x^2} + C$$

EXAMPLE 2: Find $\int \ln(x^2+2)\,dx$.

Take $u = \ln(x^2+2)$ and $dv = dx$; then $du = \dfrac{2x\,dx}{x^2+2}$ and $v = x$. By (*31.1*),

$$\int \ln(x^2+2)\,dx = x\ln(x^2+2) - \int \frac{2x^2\,dx}{x^2+2} = x\ln(x^2+2) - \int \left(2 - \frac{4}{x^2+2}\right) dx$$

$$= x\ln(x^2+2) - 2x + 2\sqrt{2}\arctan\frac{x}{\sqrt{2}} + C$$

(See Problems 1 to 10.)

REDUCTION FORMULAS. The labor involved in successive applications of integration by parts to evaluate an integral (see Problem 9) may be materially reduced by the use of *reduction formulas*. In general, a reduction formula yields a new integral of the same form as the original but with an exponent increased or reduced. A reduction formula succeeds if ultimately it produces an integral that can be evaluated. Among the reduction formulas are:

$$\int \frac{dx}{(a^2 \pm x^2)^m} = \frac{1}{a^2}\left[\frac{x}{(2m-2)(a^2 \pm x^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}\right],\quad m \neq 1 \tag{31.2}$$

$$\int (a^2 \pm x^2)^m\,dx = \frac{x(a^2 \pm x^2)^m}{2m+1} + \frac{2ma^2}{2m+1}\int (a^2 \pm x^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.3}$$

$$\int \frac{dx}{(x^2 - a^2)^m} = -\frac{1}{a^2}\left[\frac{x}{(2m-2)(x^2 - a^2)^{m-1}} + \frac{2m-3}{2m-2}\int \frac{dx}{(x^2 - a^2)^{m-1}}\right],\quad m \neq 1 \tag{31.4}$$

$$\int (x^2 - a^2)^m\,dx = \frac{x(x^2 - a^2)^m}{2m+1} - \frac{2ma^2}{2m+1}\int (x^2 - a^2)^{m-1}\,dx,\quad m \neq -1/2 \tag{31.5}$$

$$\int x^m\,e^{ax}\,dx = \frac{1}{a}x^m\,e^{ax} - \frac{m}{a}\int x^{m-1}\,e^{ax}\,dx \tag{31.6}$$

$$\int \sin^m x\,dx = -\frac{\sin^{m-1}x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2}x\,dx \tag{31.7}$$

$$\int \cos^m x\,dx = \frac{\cos^{m-1}x\sin x}{m} + \frac{m-1}{m}\int \cos^{m-2}x\,dx \tag{31.8}$$

$$\begin{aligned}\int \sin^m x\cos^n x\,dx &= \frac{\sin^{m+1}x\cos^{n-1}x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2}x\,dx\\ &= -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx\,,\quad m \neq -n\end{aligned} \tag{31.9}$$

$$\int x^m \sin bx\,dx = -\frac{x^m}{b}\cos bx + \frac{m}{b}\int x^{m-1}\cos bx\,dx \tag{31.10}$$

$$\int x^m \cos bx\,dx = \frac{x^m}{b}\sin bx - \frac{m}{b}\int x^{m-1}\sin bx\,dx \tag{31.11}$$

(See Problem 11.)

Solved Problems

1. Find $\int x\sin x\,dx$.

We have three choices: (*a*) $u = x\sin x$, $dv = dx$; (*b*) $u = \sin x$, $dv = x\,dx$; (*c*) $u = x$, $dv = \sin x\,dx$.

(*a*) Let $u = x\sin x$, $dv = dx$. Then $du = (\sin x + x\cos x)\,dx$, $v = x$, and

$$\int x\sin x\,dx = x\cdot x\sin x - \int x(\sin x + x\cos x)\,dx$$

The resulting integral is not as simple as the original, and this choice is discarded.

(*b*) Let $u = \sin x$, $dv = x\,dx$. Then $du = \cos x\,dx$, $v = \frac{1}{2}x^2$, and

$$\int x\sin x\,dx = \tfrac{1}{2}x^2\sin x - \int \tfrac{1}{2}x^2\cos x\,dx$$

The resulting integral is not as simple as the original, and this choice too is discarded.

(*c*) Let $u = x$, $dv = \sin x\,dx$. Then $du = dx$, $v = -\cos x$, and

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$$

2. Find $\int xe^x\,dx$.

Let $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C$$

3. Find $\int x^2\ln x\,dx$.

Let $u = \ln x$, $dv = x^2\,dx$. Then $du = \frac{dx}{x}$, $v = \frac{x^3}{3}$, and

$$\int x^2\ln x\,dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{dx}{x} = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + C$$

4. Find $\int x\sqrt{1+x}\,dx$.

Let $u = x$, $dv = \sqrt{1+x}\,dx$. Then $du = dx$, $v = \frac{2}{3}(1+x)^{3/2}$, and

$$\int x\sqrt{1+x}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{2}{3}\int (1+x)^{3/2}\,dx = \tfrac{2}{3}x(1+x)^{3/2} - \tfrac{4}{15}(1+x)^{5/2} + C$$

5. Find $\int \arcsin x\,dx$.

Let $u = \arcsin x$, $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$, and

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C$$

6. Find $\int \sin^2 x\,dx$.

Let $u = \sin x$, $dv = \sin x\,dx$. Then $du = \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^2 x\,dx = -\sin x\cos x + \int \cos^2 x\,dx = -\sin x\cos x + \int (1-\sin^2 x)\,dx$$

$$= -\tfrac{1}{2}\sin 2x + \int dx - \int \sin^2 x\,dx$$

Hence $\quad 2\int \sin^2 x\,dx = -\tfrac{1}{2}\sin 2x + x + C' \quad$ and $\quad \int \sin^2 x\,dx = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$

7. Find $\int \sec^3 x\,dx$.

Let $u = \sec x$, $dv = \sec^2 x\,dx$. Then $du = \sec x\tan x\,dx$, $v = \tan x$, and

$$\int \sec^3 x\,dx = \sec x\tan x - \int \sec x\tan^2 x\,dx = \sec x\tan x - \int \sec x(\sec^2 x - 1)\,dx$$

$$= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

Then $\quad 2\int \sec^3 x\,dx = \sec x\tan x + \int \sec x\,dx = \sec x\tan x + \ln|\sec x + \tan x| + C'$

and $\quad \int \sec^3 x\,dx = \tfrac{1}{2}\{\sec x\tan x + \ln|\sec x + \tan x|\} + C$

8. Find $\int x^2\sin x\,dx$.

Let $u = x^2$, $dv = \sin x\,dx$. Then $du = 2x\,dx$, $v = -\cos x$, and

$$\int x^2\sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

For the resulting integral, let $u = x$ and $dv = \cos x\,dx$. Then $du = dx$, $v = \sin x$, and

$$\int x^2\sin x\,dx = -x^2\cos x + 2\left(x\sin x - \int \sin x\,dx\right) = -x^2\cos x + 2x\sin x + 2\cos x + C$$

9. Find $\int x^3 e^{2x}\,dx$.

Let $u = x^3$, $dv = e^{2x}\,dx$. Then $du = 3x^2\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\,dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\int x^2 e^{2x}\,dx$$

For the resulting integral, let $u = x^2$ and $dv = e^{2x}\,dx$. Then $du = 2x\,dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\,dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{2}\left(\tfrac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\int x e^{2x}\,dx$$

For the resulting integral, let $u = x$ and $dv = e^{2x}\,dx$. Then $du = dx$, $v = \frac{1}{2}e^{2x}$, and

$$\int x^3 e^{2x}\,dx = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{2}\left(\tfrac{1}{2}x e^{2x} - \tfrac{1}{2}\int e^{2x}\,dx\right) = \tfrac{1}{2}x^3 e^{2x} - \tfrac{3}{4}x^2 e^{2x} + \tfrac{3}{4}x e^{2x} - \tfrac{3}{8}e^{2x} + C$$

10. Find reduction formulas for (a) $\displaystyle\int \frac{x^2\,dx}{(a^2 \pm x^2)^m}$ and (b) $\displaystyle\int x^2(a^2 \pm x^2)^{m-1}\,dx$.

(a) Take $u = x$, $dv = \dfrac{x\,dx}{(a^2 \pm x^2)^m}$; then $du = dx$, $v = \dfrac{\mp 1}{(2m-2)(a^2 \pm x^2)^{m-1}}$, and

$$\int \frac{x^2\,dx}{(a^2 \pm x^2)^m} = \frac{\mp x}{(2m-2)(a^2 \pm x^2)^{m-1}} \pm \frac{1}{2m-2}\int \frac{dx}{(a^2 \pm x^2)^{m-1}}$$

(b) Take $u = x$, $dv = x(a^2 \pm x^2)^{m-1}\,dx$; then $du = dx$, $v = \dfrac{\pm 1}{2m}(a^2 \pm x^2)^m$, and

$$\int x^2(a^2 \pm x^2)^{m-1}\,dx = \frac{\pm x}{2m}(a^2 \pm x^2)^m \mp \frac{1}{2m}\int (a^2 \pm x^2)^m\,dx$$

11. Find: (a) $\displaystyle\int \frac{dx}{(1+x^2)^{5/2}}$ and (b) $\displaystyle\int (9 + x^2)^{3/2}\,dx$.

(a) Since (31.2) reduces the exponent in the denominator by 1, we use this formula twice to obtain

$$\int \frac{dx}{(1+x^2)^{5/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{3(1+x^2)^{3/2}} + \frac{2}{3}\frac{x}{(1+x^2)^{1/2}} + C$$

(b) Using (31.3), we obtain

$$\int (9+x^2)^{3/2}\,dx = \tfrac{1}{4}x(9+x^2)^{3/2} + \tfrac{27}{4}\int (9+x^2)^{1/2}\,dx$$
$$= \tfrac{1}{4}x(9+x^2)^{3/2} + \tfrac{27}{8}[x(9+x^2)^{1/2} + 9\ln(x + \sqrt{9+x^2})] + C$$

12. Derive reduction formula (31.7): $\displaystyle\int \sin^m x\,dx = -\frac{\sin^{m-1} x\cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x\,dx.$

We use integration by parts: Let $u = \sin^{m-1} x$ and $dv = \sin x\,dx$; then $du = (m-1)\sin^{m-2} x \cos x\,dx$, $v = -\cos x$, and

$$\int \sin^m x\,dx = -\cos x\sin^{m-1} x + (m-1)\int \sin^{m-2} x\cos^2 x\,dx$$
$$= -\cos x\sin^{m-1} x + (m-1)\int (\sin^{m-2} x)(1 - \sin^2 x)\,dx$$
$$= -\cos x\sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx - (m-1)\int \sin^m x\,dx$$

Hence, $$m\int \sin^m x\,dx = -\cos x\sin^{m-1} x + (m-1)\int \sin^{m-2} x\,dx$$

and division by m yields (31.7).

Supplementary Problems

In Problems 13 to 29 and 32 to 40 evaluate the indefinite integral at left.

13. $\int x \cos x \, dx = x \sin x + \cos x + C$

14. $\int x \sec^2 3x \, dx = \frac{1}{3} x \tan 3x - \frac{1}{9} \ln|\sec 3x| + C$

15. $\int \arccos 2x \, dx = x \arccos 2x - \frac{1}{2}\sqrt{1 - 4x^2} + C$

16. $\int \arctan x \, dx = x \arctan x - \ln \sqrt{1 + x^2} + C$

17. $\int x^2 \sqrt{1 - x} \, dx = -\frac{2}{105}(1 - x)^{3/2}(15x^2 + 12x + 8) + C$

18. $\int \frac{xe^x \, dx}{(1 + x)^2} = \frac{e^x}{1 + x} + C$

19. $\int x \arctan x \, dx = \frac{1}{2}(x^2 + 1) \arctan x - \frac{1}{2}x + C$

20. $\int x^2 e^{-3x} \, dx = -\frac{1}{3} e^{-3x}(x^2 + \frac{2}{3}x + \frac{2}{9}) + C$

21. $\int \sin^3 x \, dx = -\frac{2}{3} \cos^3 x - \sin^2 x \cos x + C$

22. $\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$

23. $\int \frac{x \, dx}{\sqrt{a + bx}} = \frac{2(bx - 2a)\sqrt{a + bx}}{3b^2} + C$

24. $\int \frac{x^2 \, dx}{\sqrt{1 + x}} = \frac{2}{15}(3x^2 - 4x + 8)\sqrt{1 + x} + C$

25. $\int x \arcsin x^2 \, dx = \frac{1}{2} x^2 \arcsin x^2 + \frac{1}{2}\sqrt{1 - x^4} + C$

26. $\int \sin x \sin 3x \, dx = \frac{1}{8} \sin 3x \cos x - \frac{3}{8} \sin x \cos 3x + C$

27. $\int \sin(\ln x) \, dx = \frac{1}{2} x(\sin \ln x - \cos \ln x) + C$

28. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}(b \sin bx + a \cos bx)}{a^2 + b^2} + C$

29. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C$

30. (*a*) Write $\int \frac{a^2 \, dx}{(a^2 \pm x^2)^m} = \int \frac{(a^2 \pm x^2) \mp x^2}{(a^2 \pm x^2)^m} dx = \int \frac{dx}{(a^2 \pm x^2)^{m-1}} \mp \int \frac{x^2 \, dx}{(a^2 \pm x^2)^m}$ and use the result of Problem 10(*a*) to obtain (*31.2*).

(*b*) Write $\int (a^2 \pm x^2)^m \, dx = a^2 \int (a^2 \pm x^2)^{m-1} \, dx \pm \int x^2(a^2 \pm x^2)^{m-1} \, dx$ and use the result of Problem 10(*b*) to obtain (*31.3*).

31. Derive reduction formulas (*31.4*) to (*31.11*).

32. $\int \frac{dx}{(1 - x^2)^3} = \frac{x(5 - 3x^2)}{8(1 - x^2)^2} + \frac{3}{16} \ln \left| \frac{1 + x}{1 - x} \right| + C$

33. $\displaystyle\int \frac{dx}{(4+x^2)^{3/2}} = \frac{x}{4(4+x^2)^{1/2}} + C$

34. $\displaystyle\int (4-x^2)^{3/2}\,dx = \tfrac{1}{4}x(10-x^2)\sqrt{4-x^2} + 6\arcsin\tfrac{1}{2}x + C$

35. $\displaystyle\int \frac{dx}{(x^2-16)^3} = \frac{1}{2048}\left[\frac{x(3x^2-80)}{(x^2-16)^2} + \frac{3}{8}\ln\left|\frac{x-4}{x+4}\right|\right] + C$

36. $\displaystyle\int (x^2-1)^{5/2}\,dx = \tfrac{1}{48}x(8x^4-26x^2+33)\sqrt{x^2-1} - \tfrac{5}{16}\ln|x+\sqrt{x^2-1}| + C$

37. $\displaystyle\int \sin^4 x\,dx = \tfrac{3}{8}x - \tfrac{3}{8}\sin x\cos x - \tfrac{1}{4}\sin^3 x\cos x + C$

38. $\displaystyle\int \cos^5 x\,dx = \tfrac{1}{15}(3\cos^4 x + 4\cos^2 x + 8)\sin x + C$

39. $\displaystyle\int \sin^3 x\cos^2 x\,dx = -\tfrac{1}{5}\cos^3 x\,(\sin^2 x + \tfrac{2}{3}) + C$

40. $\displaystyle\int \sin^4 x\cos^5 x\,dx = \tfrac{1}{9}\sin^5 x\,(\cos^4 x + \tfrac{4}{7}\cos^2 x + \tfrac{8}{35}) + C$

An alternative procedure for some of the more tedious problems of this section can be found by noting (see Problem 9) that in

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{3}{4}xe^{2x} - \tfrac{3}{8}e^{2x} + C \tag{1}$$

the terms on the right, apart from the coefficients, are the different terms obtained by repeated differentiations of the integrand x^3e^{2x}. Thus, we may write at once

$$\int x^3e^{2x}\,dx = Ax^3e^{2x} + Bx^2e^{2x} + Dxe^{2x} + Ee^{2x} + C \tag{2}$$

and from it obtain by differentiation

$$x^3e^{2x} = 2Ax^3e^{2x} + (3A+2B)x^2e^{2x} + (2B+2D)xe^{2x} + (D+2E)e^{2x}$$

Equating coefficients, we have

$$2A = 1 \qquad 3A+2B=0 \qquad 2B+2D=0 \qquad D+2E=0$$

so that $A=\tfrac{1}{2}$, $B=-\tfrac{3}{2}A=-\tfrac{3}{4}$, $D=-B=\tfrac{3}{4}$, $E=-\tfrac{1}{2}D=-\tfrac{3}{8}$. Substituting for A, B, D, E in (2), we obtain (1).

This procedure may be used for finding $\int f(x)\,dx$ whenever repeated differentiation of $f(x)$ yields only a finite number of different terms.

41. Find $\displaystyle\int e^{2x}\cos 3x\,dx = \tfrac{1}{13}e^{2x}(3\sin 3x + 2\cos 3x) + C$, using

$$\int e^{2x}\cos 3x\,dx = Ae^{2x}\sin 3x + Be^{2x}\cos 3x + C$$

42. Find $\displaystyle\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = \tfrac{1}{25}e^{3x}(-14\sin 4x - 23\cos 4x) + C$, using

$$\int e^{3x}(2\sin 4x - 5\cos 4x)\,dx = Ae^{3x}\sin 4x + Be^{3x}\cos 4x + C$$

43. Find $\displaystyle\int \sin 3x\cos 2x\,dx = -\tfrac{1}{5}(2\sin 3x\sin 2x + 3\cos 3x\cos 2x) + C$, using

$$\int \sin 3x\cos 2x\,dx = A\sin 3x\sin 2x + B\cos 3x\cos 2x + D\cos 3x\sin 2x + E\sin 3x\cos 2x + C$$

44. Find $\displaystyle\int e^{3x}x^2\sin x\,dx = \frac{e^{3x}}{250}[25x^2(3\sin x - \cos x) - 10x(4\sin x - 3\cos x) + 9\sin x - 13\cos x] + C.$

Chapter 32

Trigonometric Integrals

THE FOLLOWING IDENTITIES are employed to find some of the trigonometric integrals of this chapter:

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin x \cos x = \frac{1}{2} \sin 2x$
7. $\sin x \cos y = \frac{1}{2}[\sin(x - y) + \sin(x + y)]$
8. $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$
10. $1 - \cos x = 2 \sin^2 \frac{1}{2}x$
11. $1 + \cos x = 2 \cos^2 \frac{1}{2}x$
12. $1 \pm \sin x = 1 \pm \cos(\frac{1}{2}\pi - x)$

TWO SPECIAL SUBSTITUTION RULES are useful in a few simple cases:

1. For $\int \sin^m x \cos^n x\, dx$: If m is odd, substitute $u = \cos x$. If n is odd, substitute $u = \sin x$.
2. For $\int \tan^m x \sec^n x\, dx$: If n is even, substitute $u = \tan x$. If m is odd, substitute $u = \sec x$.

Solved Problems

SINES AND COSINES

In Problems 1 to 17, evaluate the integral at the left.

1. $\int \sin^2 x\, dx = \int \frac{1}{2}(1 - \cos 2x)\, dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

2. $\int \cos^2 3x\, dx = \int \frac{1}{2}(1 + \cos 6x)\, dx = \frac{1}{2}x + \frac{1}{12}\sin 6x + C$

3. $\int \sin^3 x\, dx = \int \sin^2 x \sin x\, dx = \int (1 - \cos^2 x) \sin x\, dx = -\cos x + \frac{1}{3}\cos^3 x + C$

This solution is equivalent to using the substitution $u = \cos x$, $du = -\sin x\, dx$, as follows:

$$\int \sin^3 x\, dx = -\int (1 - u^2)\, du = -u + \frac{1}{3}u^3 + C = -\cos x + \frac{1}{3}\cos^3 x + C$$

4. $\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2 \cos x\, dx$

$$= \int \cos x\, dx - 2\int \sin^2 x \cos x\, dx + \int \sin^4 x \cos x\, dx$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

This amounts to the use of the substitution $u = \sin x$. We have also used (*30.2*).

5. $$\int \sin^2 x \cos^3 x\, dx = \int \sin^2 x \cos^2 x \cos x\, dx = \int \sin^2 x\,(1 - \sin^2 x) \cos x\, dx$$
$$= \int \sin^2 x \cos x\, dx - \int \sin^4 x \cos x\, dx = \tfrac{1}{3}\sin^3 x - \tfrac{1}{5}\sin^5 x + C$$

6. $$\int \cos^4 2x \sin^3 2x\, dx = \int \cos^4 2x \sin^2 2x \sin 2x\, dx = \int \cos^4 2x\,(1 - \cos^2 2x) \sin 2x\, dx$$
$$= \int \cos^4 2x \sin 2x\, dx - \int \cos^6 2x \sin 2x\, dx = -\tfrac{1}{10}\cos^5 2x + \tfrac{1}{14}\cos^7 2x + C$$

7. $$\int \sin^3 3x \cos^5 3x\, dx = \int (1 - \cos^2 3x) \cos^5 3x \sin 3x\, dx$$
$$= \int \cos^5 3x \sin 3x\, dx - \int \cos^7 3x \sin 3x\, dx = -\tfrac{1}{18}\cos^6 3x + \tfrac{1}{24}\cos^8 3x + C$$

or $$\int \sin^3 3x \cos^5 3x\, dx = \int \sin^3 3x\,(1 - \sin^2 3x)^2 \cos 3x\, dx$$
$$= \int \sin^3 3x \cos 3x\, dx - 2\int \sin^5 3x \cos 3x\, dx + \int \sin^7 3x \cos 3x\, dx$$
$$= \tfrac{1}{12}\sin^4 3x - \tfrac{1}{9}\sin^6 3x + \tfrac{1}{24}\sin^8 3x + C$$

8. $$\int \cos^3 \frac{x}{3}\, dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cos \frac{x}{3}\, dx = 3\sin \frac{x}{3} - \sin^3 \frac{x}{3} + C$$

9. $$\int \sin^4 x\, dx = \int (\sin^2 x)^2\, dx = \tfrac{1}{4}\int (1 - \cos 2x)^2\, dx$$
$$= \tfrac{1}{4}\int dx - \tfrac{1}{2}\int \cos 2x\, dx + \tfrac{1}{4}\int \cos^2 2x\, dx$$
$$= \tfrac{1}{4}\int dx - \tfrac{1}{2}\int \cos 2x\, dx + \tfrac{1}{8}\int (1 + \cos 4x)\, dx$$
$$= \tfrac{1}{4}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{8}x + \tfrac{1}{32}\sin 4x + C = \tfrac{3}{8}x - \tfrac{1}{4}\sin 2x + \tfrac{1}{32}\sin 4x + C$$

10. $$\int \sin^2 x \cos^2 x\, dx = \tfrac{1}{4}\int \sin^2 2x\, dx = \tfrac{1}{8}\int (1 - \cos 4x)\, dx = \tfrac{1}{8}x - \tfrac{1}{32}\sin 4x + C$$

11. $$\int \sin^4 3x \cos^2 3x\, dx = \int (\sin^2 3x \cos^2 3x) \sin^2 3x\, dx = \tfrac{1}{8}\int \sin^2 6x\,(1 - \cos 6x)\, dx$$
$$= \tfrac{1}{8}\int \sin^2 6x\, dx - \tfrac{1}{8}\int \sin^2 6x \cos 6x\, dx$$
$$= \tfrac{1}{16}\int (1 - \cos 12x)\, dx - \tfrac{1}{8}\int \sin^2 6x \cos 6x\, dx$$
$$= \tfrac{1}{16}x - \tfrac{1}{192}\sin 12x - \tfrac{1}{144}\sin^3 6x + C$$

12. $$\int \sin 3x \sin 2x\, dx = \int \tfrac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)]\, dx = \tfrac{1}{2}\int (\cos x - \cos 5x)\, dx$$
$$= \tfrac{1}{2}\sin x - \tfrac{1}{10}\sin 5x + C$$

13. $$\int \sin 3x \cos 5x\, dx = \int \tfrac{1}{2}[\sin(3x - 5x) + \sin(3x + 5x)]\, dx = \tfrac{1}{4}\cos 2x - \tfrac{1}{16}\cos 8x + C$$

14. $\int \cos 4x \cos 2x \, dx = \frac{1}{2}\int (\cos 2x + \cos 6x)\, dx = \frac{1}{4}\sin 2x + \frac{1}{12}\sin 6x + C$

15. $\int \sqrt{1-\cos x}\, dx = \sqrt{2}\int \sin \frac{1}{2}x \, dx = -2\sqrt{2}\cos \frac{1}{2}x + C$

16. $\int (1+\cos 3x)^{3/2}\, dx = 2\sqrt{2}\int \cos^3 \frac{3}{2}x\, dx = 2\sqrt{2}\int (1-\sin^2 \frac{3}{2}x)\cos \frac{3}{2}x\, dx$

$$= 2\sqrt{2}\left(\tfrac{2}{3}\sin \tfrac{3}{2}x - \tfrac{2}{9}\sin^3 \tfrac{3}{2}x\right) + C$$

17. $\int \frac{dx}{\sqrt{1-\sin 2x}} = \int \frac{dx}{\sqrt{1-\cos(\frac{1}{2}\pi - 2x)}} = \frac{\sqrt{2}}{2}\int \frac{dx}{\sin(\frac{1}{4}\pi - x)} = \frac{\sqrt{2}}{2}\int \csc(\frac{1}{4}\pi - x)\, dx$

$$= -\frac{\sqrt{2}}{2}\ln\left|\csc(\tfrac{1}{4}\pi - x) - \cot(\tfrac{1}{4}\pi - x)\right| + C$$

TANGENTS, SECANTS, COTANGENTS, COSECANTS

Evaluate the integral at the left.

18. $\int \tan^4 x\, dx = \int \tan^2 x \tan^2 x\, dx = \int \tan^2 x(\sec^2 x - 1)\, dx = \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$

$$= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$$

19. $\int \tan^5 x\, dx = \int \tan^3 x \tan^2 x\, dx = \int \tan^3 x(\sec^2 x - 1)\, dx$

$$= \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx = \int \tan^3 x \sec^2 x\, dx - \int \tan x(\sec^2 x - 1)\, dx$$

$$= \tfrac{1}{4}\tan^4 x - \tfrac{1}{2}\tan^2 x + \ln|\sec x| + C$$

20. $\int \sec^4 2x\, dx = \int \sec^2 2x \sec^2 2x\, dx = \int \sec^2 2x(1+\tan^2 2x)\, dx$

$$= \int \sec^2 2x\, dx + \int \tan^2 2x \sec^2 2x\, dx = \tfrac{1}{2}\tan 2x + \tfrac{1}{6}\tan^3 2x + C$$

21. $\int \tan^3 3x \sec^4 3x\, dx = \int \tan^3 3x(1+\tan^2 3x)\sec^2 3x\, dx$

$$= \int \tan^3 3x \sec^2 3x\, dx + \int \tan^5 3x \sec^2 3x\, dx = \tfrac{1}{12}\tan^4 3x + \tfrac{1}{18}\tan^6 3x + C$$

22. $\int \tan^2 x \sec^3 x\, dx = \int (\sec^2 x - 1)\sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx$

$$= \tfrac{1}{4}\sec^3 x \tan x - \tfrac{1}{8}\sec x \tan x - \tfrac{1}{8}\ln|\sec x + \tan x| + C \quad \text{(integrating by parts)}$$

23. $\int \tan^3 2x \sec^3 2x\, dx = \int (\tan^2 2x \sec^2 2x)(\sec 2x \tan 2x\, dx)$

$$= \int (\sec^2 2x - 1)(\sec^2 2x)(\sec 2x \tan 2x\, dx)$$

$$= \int (\sec^4 2x)(\sec 2x \tan 2x\, dx) - \int (\sec^2 2x)(\sec 2x \tan 2x\, dx)$$

$$= \tfrac{1}{10}\sec^5 2x - \tfrac{1}{6}\sec^3 2x + C$$

24. $\int \cot^3 2x\,dx = \int \cot 2x\,(\csc^2 2x - 1)\,dx = -\frac{1}{4}\cot^2 2x + \frac{1}{2}\ln|\csc 2x| + C$

25. $\int \cot^4 3x\,dx = \int \cot^2 3x\,(\csc^2 3x - 1)\,dx = \int \cot^2 3x \csc^2 3x\,dx - \int \cot^2 3x\,dx$

$$= \int \cot^2 3x \csc^2 3x\,dx - \int (\csc^2 3x - 1)\,dx = -\frac{1}{9}\cot^3 3x + \frac{1}{3}\cot 3x + x + C$$

26. $\int \csc^6 x\,dx = \int \csc^2 x(1 + \cot^2 x)^2\,dx = \int \csc^2 x\,dx + 2\int \cot^2 x \csc^2 x\,dx + \int \cot^4 x \csc^2 x\,dx$

$$= -\cot x - \frac{2}{3}\cot^3 x - \frac{1}{5}\cot^5 x + C$$

27. $\int \cot 3x \csc^4 3x\,dx = \int \cot 3x\,(1 + \cot^2 3x)\csc^2 3x\,dx$

$$= \int \cot 3x \csc^2 3x\,dx + \int \cot^3 3x \csc^2 3x\,dx = -\frac{1}{6}\cot^2 3x - \frac{1}{12}\cot^4 3x + C$$

28. $\int \cot^3 x \csc^5 x\,dx = \int (\cot^2 x \csc^4 x)(\csc x \cot x\,dx) = \int (\csc^2 x - 1)(\csc^4 x)(\csc x \cot x\,dx)$

$$= \int (\csc^6 x)(\csc x \cot x\,dx) - \int (\csc^4 x)(\csc x \cot x\,dx) = -\frac{1}{7}\csc^7 x + \frac{1}{5}\csc^5 x + C$$

Supplementary Problems

In Problems 29 to 56, evaluate the integral at the left.

29. $\int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

30. $\int \sin^3 2x\,dx = \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$

31. $\int \sin^4 2x\,dx = \frac{3}{8}x - \frac{1}{8}\sin 4x + \frac{1}{64}\sin 8x + C$

32. $\int \cos^4 \frac{1}{2}x\,dx = \frac{3}{8}x + \frac{1}{2}\sin x + \frac{1}{16}\sin 2x + C$

33. $\int \sin^7 x\,dx = \frac{1}{7}\cos^7 x - \frac{3}{5}\cos^5 x + \cos^3 x - \cos x + C$

34. $\int \cos^6 \frac{1}{2}x\,dx = \frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C$

35. $\int \sin^2 x \cos^5 x\,dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

36. $\int \sin^3 x \cos^2 x\,dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

37. $\int \sin^3 x \cos^3 x\,dx = \frac{1}{48}\cos^3 2x - \frac{1}{16}\cos 2x + C$

38. $\int \sin^4 x \cos^4 x\,dx = \frac{1}{128}(3x - \sin 4x + \frac{1}{8}\sin 8x) + C$

39. $\int \sin 2x \cos 4x\,dx = \frac{1}{4}\cos 2x - \frac{1}{12}\cos 6x + C$

40. $\int \cos 3x \cos 2x\,dx = \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C$

41. $\int \sin 5x \sin x\,dx = \frac{1}{8}\sin 4x - \frac{1}{12}\sin 6x + C$

42. $\int \frac{\cos^3 x\,dx}{1 - \sin x} = \sin x + \frac{1}{2}\sin^2 x + C$

43. $\int \frac{\cos^{2/3} x}{\sin^{8/3} x}\,dx = -\frac{3}{5}\cot^{5/3} x + C$

44. $\int \frac{\cos^3 x}{\sin^4 x}\,dx = \csc x - \frac{1}{3}\csc^3 x + C$

45. $\int x(\cos^3 x^2 - \sin^3 x^2)\,dx = \frac{1}{12}(\sin x^2 + \cos x^2)(4 + \sin 2x^2) + C$

46. $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$

47. $\int \tan^3 3x \sec 3x\,dx = \frac{1}{9}\sec^3 3x - \frac{1}{3}\sec 3x + C$

48. $\int \tan^{3/2} x \sec^4 x\,dx = \frac{2}{5}\tan^{5/2} x + \frac{2}{9}\tan^{9/2} x + C$

49. $\int \tan^4 x \sec^4 x\,dx = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + C$

50. $\int \csc^4 2x\,dx = -\frac{1}{2}\cot 2x - \frac{1}{6}\cot^3 2x + C$

51. $\int \cot^3 x\,dx = -\frac{1}{2}\cot^2 x - \ln|\sin x| + C$

52. $\int \left(\frac{\sec x}{\tan x}\right)^4 dx = -\frac{1}{3\tan^3 x} - \frac{1}{\tan x} + C$

53. $\int \cot^3 x \csc^4 x\,dx = -\frac{1}{4}\cot^4 x - \frac{1}{6}\cot^6 x + C$

54. $\int \frac{\cot^3 x}{\csc x}\,dx = -\sin x - \csc x + C$

55. $\int \cot^3 x \csc^3 x\,dx = -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + C$

56. $\int \tan x \sqrt{\sec x}\,dx = 2\sqrt{\sec x} + C$

57. Use integration by parts to derive the reduction formulas

$$\int \sec^m u\,du = \frac{1}{m-1}\sec^{m-2} u \tan u + \frac{m-2}{m-1}\int \sec^{m-2} u\,du$$

and
$$\int \csc^m u\,du = -\frac{1}{m-1}\csc^{m-2} u \cot u + \frac{m-2}{m-1}\int \csc^{m-2} u\,du$$

Use the reduction formulas of Problem 57 to evaluate the left-hand integral in Problems 58 to 60.

58. $\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

59. $\int \csc^5 x\,dx = -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x + \frac{3}{8}\ln|\csc x - \cot x| + C$

60. $\int \sec^6 x\,dx = \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x + C = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

Chapter 33

Trigonometric Substitutions

SOME INTEGRATIONS may be simplified with the following substitutions:

1. If an integrand contains $\sqrt{a^2 - x^2}$, substitute $x = a \sin z$.
2. If an integrand contains $\sqrt{a^2 + x^2}$, substitute $x = a \tan z$.
3. If an integrand contains $\sqrt{x^2 - a^2}$, substitute $x = a \sec z$.

More generally, an integrand that contains one of the forms $\sqrt{a^2 - b^2x^2}$, $\sqrt{a^2 + b^2x^2}$, or $\sqrt{b^2x^2 - a^2}$ but no other irrational factor may be transformed into another involving trigonometric functions of a new variable as follows:

For	Use	To obtain
$\sqrt{a^2 - b^2x^2}$	$x = \dfrac{a}{b} \sin z$	$a\sqrt{1 - \sin^2 z} = a \cos z$
$\sqrt{a^2 + b^2x^2}$	$x = \dfrac{a}{b} \tan z$	$a\sqrt{1 + \tan^2 z} = a \sec z$
$\sqrt{b^2x^2 - a^2}$	$x = \dfrac{a}{b} \sec z$	$a\sqrt{\sec^2 z - 1} = a \tan z$

In each case, integration yields an expression in the variable z. The corresponding expression in the original variable may be obtained by the use of a right triangle as shown in the solved problems that follow.

Solved Problems

1. Find $\displaystyle\int \frac{dx}{x^2\sqrt{4 + x^2}}$.

Let $x = 2\tan z$, so that x and z are related as in Fig. 33-1. Then $dx = 2\sec^2 z\, dz$ and $\sqrt{4 + x^2} = 2\sec z$, and

$$\int \frac{dx}{x^2\sqrt{4 + x^2}} = \int \frac{2\sec^2 z\, dz}{(4\tan^2 z)(2\sec z)} = \frac{1}{4}\int \frac{\sec z}{\tan^2 z}\, dz = \frac{1}{4}\int \sin^{-2} z \cos z\, dz$$

$$= -\frac{1}{4\sin z} + C = -\frac{\sqrt{4 + x^2}}{4x} + C$$

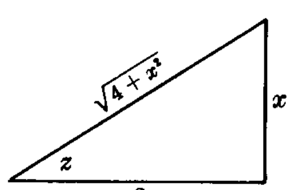

Fig. 33-1

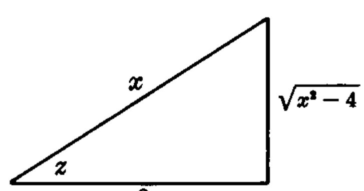

Fig. 33-2

2. Find $\int \frac{x^2}{\sqrt{x^2-4}}\,dx$.

Let $x = 2\sec z$, so that x and z are related as in Fig. 33-2. Then $dx = 2\sec z \tan z\,dz$ and $\sqrt{x^2-4} = 2\tan z$, and

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2-4}}\,dx &= \int \frac{4\sec^2 z}{2\tan z}(2\sec z\tan z\,dz) = 4\int \sec^3 z\,dz\\ &= 2\sec z\tan z + 2\ln|\sec z + \tan z| + C'\\ &= \tfrac{1}{2}x\sqrt{x^2-4} + 2\ln|x + \sqrt{x^2-4}| + C\end{aligned}$$

3. Find $\int \frac{\sqrt{9-4x^2}}{x}\,dx$.

Let $x = \frac{3}{2}\sin z$ (see Fig. 33-3); then $dx = \frac{3}{2}\cos z\,dz$ and $\sqrt{9-4x^2} = 3\cos z$, and

$$\begin{aligned}\int \frac{\sqrt{9-4x^2}}{x}\,dx &= \int \frac{3\cos z}{\frac{3}{2}\sin z}\left(\frac{3}{2}\cos z\,dz\right) = 3\int \frac{\cos^2 z}{\sin z}\,dz = 3\int \frac{1-\sin^2 z}{\sin z}\,dx\\ &= 3\int \csc z\,dz - 3\int \sin z\,dz = 3\ln|\csc z - \cot z| + 3\cos z + C'\\ &= 3\ln\left|\frac{3-\sqrt{9-4x^2}}{x}\right| + \sqrt{9-4x^2} + C\end{aligned}$$

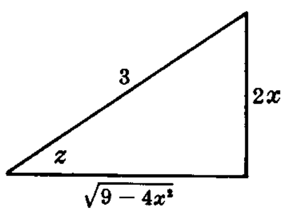

Fig. 33-3

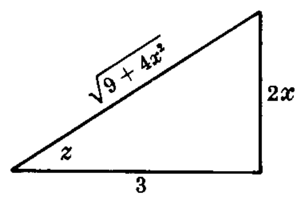

Fig. 33-4

4. Find $\int \frac{dx}{x\sqrt{9+4x^2}}$.

Let $x = \frac{3}{2}\tan z$ (see Fig. 33-4); then $dx = \frac{3}{2}\sec^2 z\,dz$ and $\sqrt{9+4x^2} = 3\sec z$, and

$$\begin{aligned}\int \frac{dx}{x\sqrt{9+4x^2}} &= \int \frac{\frac{3}{2}\sec^2 z\,dz}{(\frac{3}{2}\tan z)(3\sec z)} = \frac{1}{3}\int \csc z\,dz = \frac{1}{3}\ln|\csc z - \cot z| + C'\\ &= \frac{1}{3}\ln\left|\frac{\sqrt{9+4x^2}-3}{x}\right| + C\end{aligned}$$

5. Find $\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx$.

Let $x = \frac{4}{3}\sin z$ (see Fig. 33-5); then $dx = \frac{4}{3}\cos z\,dz$ and $\sqrt{16-9x^2} = 4\cos z$, and

$$\begin{aligned}\int \frac{(16-9x^2)^{3/2}}{x^6}\,dx &= \int \frac{(64\cos^3 z)(\frac{4}{3}\cos z\,dz)}{\frac{4096}{729}\sin^6 z} = \frac{243}{16}\int \frac{\cos^4 z}{\sin^6 z}\,dz = \frac{243}{16}\int \cot^4 z\csc^2 z\,dz\\ &= -\frac{243}{80}\cot^5 z + C = -\frac{243}{80}\,\frac{(16-9x^2)^{5/2}}{243x^5} + C = -\frac{1}{80}\,\frac{(16-9x^2)^{5/2}}{x^5} + C\end{aligned}$$

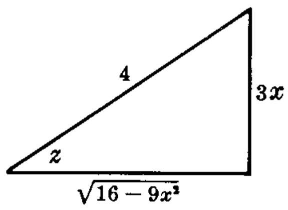

Fig. 33-5

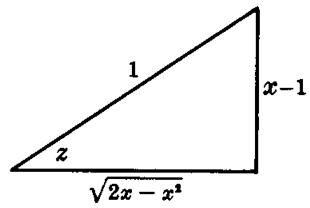

Fig. 33-6

6. Find $\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{x^2\,dx}{\sqrt{1-(x-1)^2}}$.

Let $x-1=\sin z$ (see Fig. 33-6); then $dx=\cos z\,dz$ and $\sqrt{2x-x^2}=\cos z$, and

$$\int \frac{x^2\,dx}{\sqrt{2x-x^2}} = \int \frac{(1+\sin z)^2}{\cos z}\cos z\,dz = \int (1+\sin z)^2\,dz = \int\left(\frac{3}{2}+2\sin z-\frac{1}{2}\cos 2z\right)dz$$

$$= \frac{3}{2}z - 2\cos z - \frac{1}{4}\sin 2z + C = \frac{3}{2}\arcsin(x-1) - 2\sqrt{2x-x^2} - \frac{1}{2}(x-1)\sqrt{2x-x^2}+C$$

$$= \frac{3}{2}\arcsin(x-1) - \frac{1}{2}(x+3)\sqrt{2x-x^2}+C$$

7. Find $\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{dx}{[4(x-3)^2-9]^{3/2}}$.

Let $x-3=\frac{3}{2}\sec z$ (see Fig. 33-7); then $dx=\frac{3}{2}\sec z\tan z\,dz$ and $\sqrt{4x^2-24x+27}=3\tan z$, and

$$\int \frac{dx}{(4x^2-24x+27)^{3/2}} = \int \frac{\frac{3}{2}\sec z\tan z\,dz}{27\tan^3 z} = \frac{1}{18}\int \sin^{-2} z\cos z\,dz$$

$$= -\frac{1}{18}\csc z + C = -\frac{1}{9}\frac{x-3}{\sqrt{4x^2-24x+27}}+C$$

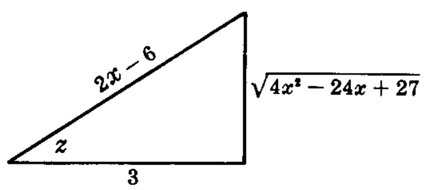

Fig. 33-7

Supplementary Problems

In Problems 8 to 22, integrate to obtain the given result.

8. $\int \frac{dx}{(4-x^2)^{3/2}} = \frac{x}{4\sqrt{4-x^2}}+C$

9. $\int \frac{\sqrt{25-x^2}}{x}\,dx = 5\ln\left|\frac{5-\sqrt{25-x^2}}{x}\right| + \sqrt{25-x^2}+C$

10. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

11. $\displaystyle\int \sqrt{x^2+4}\,dx = \tfrac{1}{2}x\sqrt{x^2+4} + 2\ln(x+\sqrt{x^2+4}) + C$

12. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\frac{x}{a} + C$

13. $\displaystyle\int \sqrt{x^2-4}\,dx = \tfrac{1}{2}x\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + C$

14. $\displaystyle\int \frac{\sqrt{x^2+a^2}}{x}\,dx = \sqrt{x^2+a^2} + \frac{a}{2}\ln\frac{\sqrt{a^2+x^2}-a}{\sqrt{a^2+x^2}+a} + C$

15. $\displaystyle\int \frac{x^2\,dx}{(4-x^2)^{5/2}} = \frac{x^3}{12(4-x^2)^{3/2}} + C$

16. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$

17. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}} = -\frac{\sqrt{9-x^2}}{9x} + C$

18. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2-16}} = \frac{1}{2}\,x\sqrt{x^2-16} + 8\ln|x+\sqrt{x^2-16}| + C$

19. $\displaystyle\int x^3\sqrt{a^2-x^2}\,dx = \frac{1}{5}\,(a^2-x^2)^{5/2} - \frac{a^2}{3}\,(a^2-x^2)^{3/2} + C$

20. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+13}} = \ln(x-2+\sqrt{x^2-4x+13}) + C$

21. $\displaystyle\int \frac{dx}{(4x-x^2)^{3/2}} = \frac{x-2}{4\sqrt{4x-x^2}} + C$

22. $\displaystyle\int \frac{dx}{(9+x^2)^2} = \frac{1}{54}\arctan\frac{x}{3} + \frac{x}{18(9+x^2)} + C$

In Problems 23 and 24, integrate by parts and apply the method of this chapter.

23. $\displaystyle\int x\arcsin x\,dx = \tfrac{1}{4}(2x^2-1)\arcsin x + \tfrac{1}{4}x\sqrt{1-x^2} + C$

24. $\displaystyle\int x\arccos x\,dx = \tfrac{1}{4}(2x^2-1)\arccos x - \tfrac{1}{4}x\sqrt{1-x^2} + C$

Chapter 34

Integration by Partial Fractions

A POLYNOMIAL IN x is a function of the form $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, where the a's are constants, $a_0 \neq 0$, and n, called the *degree* of the polynomial, is a nonnegative integer.

If two polynomials of the same degree are equal for all values of the variable, then the coefficients of the like powers of the variable in the two polynomials are equal.

Every polynomial with real coefficients can be expressed (at least, theoretically) as a product of real linear factors of the form $ax + b$ and real irreducible quadratic factors of the form $ax^2 + bx + c$. (A polynomial of degree 1 or greater is said to be *irreducible* if it cannot be factored into polynomials of lower degree.) By the quadratic formula, $ax^2 + bx + c$ is irreducible if and only if $b^2 - 4ac < 0$. (In that case, the roots of $ax^2 + bx + c = 0$ are not real.)

EXAMPLE 1: (*a*) $x^2 - x + 1$ is irreducible, since $(-1)^2 - 4(1)(1) = -3 < 0$.
(*b*) $x^2 - x - 1$ is not irreducible, since $(-1)^2 - 4(1)(-1) = 5 > 0$. In fact, $x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right)\left(x - \frac{1-\sqrt{5}}{2}\right)$.

A FUNCTION $F(x) = f(x)/g(x)$, where $f(x)$ and $g(x)$ are polynomials, is called a *rational fraction*.

If the degree of $f(x)$ is less than the degree of $g(x)$, $F(x)$ is called *proper*; otherwise, $F(x)$ is called *improper*.

An improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction. Thus, $\dfrac{x^3}{x^2+1} = x - \dfrac{x}{x^2+1}$.

Every proper rational fraction can be expressed (at least, theoretically) as a sum of simpler fractions (*partial fractions*) whose denominators are of the form $(ax + b)^n$ and $(ax^2 + bx + c)^n$, n being a positive integer. Four cases, depending upon the nature of the factors of the denominator, arise.

CASE I: DISTINCT LINEAR FACTORS. To each linear factor $ax + b$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{A}{ax+b}$, where A is a constant to be determined. (See Problems 1 and 2.)

CASE II: REPEATED LINEAR FACTORS. To each linear factor $ax + b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$$

where the A's are constants to be determined. (See Problems 3 and 4.)

CASE III: DISTINCT QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring once in the denominator of a proper rational fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined. (See Problems 5 and 6.)

CASE IV: REPEATED QUADRATIC FACTORS. To each irreducible quadratic factor $ax^2 + bx + c$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}$$

where the A's and B's are constants to be determined. (See Problems 7 and 8.)

Solved Problems

1. Find $\int \frac{dx}{x^2 - 4}$.

We factor the denominator into $(x-2)(x+2)$ and write $\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$. Clearing of fractions yields

$$1 = A(x+2) + B(x-2) \tag{1}$$

or

$$1 = (A+B)x + (2A - 2B) \tag{2}$$

We can determine the constants by either of two methods.

General method: Equate coefficients of like powers of x in (*2*) and solve simultaneously for the constants. Thus, $A + B = 0$ and $2A - 2B = 1$; $A = \frac{1}{4}$ and $B = -\frac{1}{4}$.

Short method: Substitute in (*1*) the values $x = 2$ and $x = -2$ to obtain $1 = 4A$ and $1 = -4B$; then $A = \frac{1}{4}$ and $B = -\frac{1}{4}$, as before. (Note that the values of x used are those for which the denominators of the partial fractions become 0.)

By either method, we have $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$. Then

$$\int \frac{dx}{x^2-4} = \frac{1}{4}\int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+2} = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

2. Find $\int \frac{(x+1)\,dx}{x^3 + x^2 - 6x}$.

Factoring yields $x^3 + x^2 - 6x = x(x-2)(x+3)$. Then $\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$ and

$$x + 1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2) \tag{1}$$

$$x + 1 = (A + B + C)x^2 + (A + 3B - 2C)x - 6A \tag{2}$$

General method: We solve simultaneously the system of equations

$$A + B + C = 0 \qquad A + 3B - 2C = 1 \qquad -6A = 1$$

to obtain $A = -\frac{1}{6}$, $B = \frac{3}{10}$, and $C = -\frac{2}{15}$.

Short method: We substitute in (*1*) the values $x = 0$, $x = 2$, and $x = -3$ to obtain $1 = -6A$ or $A = -1/6$, $3 = 10B$ or $B = 3/10$, and $-2 = 15C$ or $C = -2/15$.

By either method,

$$\int \frac{(x+1)\,dx}{x^3+x^2-6x} = -\frac{1}{6}\int \frac{dx}{x} + \frac{3}{10}\int \frac{dx}{x-2} - \frac{2}{15}\int \frac{dx}{x+3}$$

$$= -\frac{1}{6}\ln|x| + \frac{3}{10}\ln|x-2| - \frac{2}{15}\ln|x+3| + C = \ln\frac{|x-2|^{3/10}}{|x|^{1/6}|x+3|^{2/15}} + C$$

3. Find $\int \frac{(3x+5)\,dx}{x^3 - x^2 - x + 1}$.

$x^3 - x^2 - x + 1 = (x+1)(x-1)^2$. Hence, $\frac{3x+5}{x^3 - x^2 - x + 1} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ and

$$3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

For $x = -1$, $2 = 4A$ and $A = \frac{1}{2}$. For $x = 1$, $8 = 2C$ and $C = 4$. To determine the remaining constant, we use any other value of x, say $x = 0$; for $x = 0$, $5 = A - B + C$ and $B = -\frac{1}{2}$. Thus,

$$\int \frac{3x+5}{x^3 - x^2 - x + 1}\,dx = \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x-1} + 4\int \frac{dx}{(x-1)^2}$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| - \frac{4}{x-1} + C = -\frac{4}{x-1} + \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| + C$$

4. Find $\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx$.

The integrand is an improper fraction. By division,

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x+1}{x^3 - x^2} = x - \frac{x+1}{x^2(x-1)}$$

We write $\frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ and obtain

$$x + 1 = Ax(x-1) + B(x-1) + Cx^2$$

For $x = 0$, $1 = -B$ and $B = -1$. For $x = 1$, $2 = C$. For $x = 2$, $3 = 2A + B + 4C$ and $A = -2$. Thus,

$$\int \frac{x^4 - x^3 - x - 1}{x^3 - x^2}\,dx = \int x\,dx + 2\int \frac{dx}{x} + \int \frac{dx}{x^2} - 2\int \frac{dx}{x-1}$$

$$= \frac{1}{2}x^2 + 2\ln|x| - \frac{1}{x} - 2\ln|x-1| + C = \frac{1}{2}x^2 - \frac{1}{x} + 2\ln\left|\frac{x}{x-1}\right| + C$$

5. Find $\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx$.

$x^4 + 3x^2 + 2 = (x^2+1)(x^2+2)$. We write $\frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$ and obtain

$$x^3 + x^2 + x + 2 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$
$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

Hence $A + C = 1$, $B + D = 1$, $2A + C = 1$, and $2B + D = 2$. Solving simultaneously yields $A = 0$, $B = 1$, $C = 1$, $D = 0$. Thus,

$$\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x\,dx}{x^2+2} = \arctan x + \frac{1}{2}\ln(x^2+2) + C$$

6. Solve the equation $\int \frac{x^2\,dx}{a^4 - x^4} = \int k\,dt$, which occurs in physical chemistry.

We write $\frac{x^2}{a^4 - x^4} = \frac{A}{a-x} + \frac{B}{a+x} + \frac{Cx+D}{a^2+x^2}$. Then

$$x^2 = A(a+x)(a^2+x^2) + B(a-x)(a^2+x^2) + (Cx+D)(a-x)(a+x)$$

For $x = a$, $a^2 = 4Aa^3$ and $A = 1/4a$. For $x = -a$, $a^2 = 4Ba^3$ and $B = 1/4a$. For $x = 0$, $0 = Aa^3 + Ba^3 + Da^2 = a^2/2 + Da^2$ and $D = -\frac{1}{2}$. For $x = 2a$, $4a^2 = 15Aa^3 - 5Ba^3 - 6Ca^3 - 3Da^2$ and $C = 0$. Thus,

$$\int \frac{x^2\,dx}{a^4 - x^4} = \frac{1}{4a}\int \frac{dx}{a-x} + \frac{1}{4a}\int \frac{dx}{a+x} - \frac{1}{2}\int \frac{dx}{a^2+x^2}$$

$$= -\frac{1}{4a}\ln|a-x| + \frac{1}{4a}\ln|a+x| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

so that $$\int k\,dt = kt = \frac{1}{4a}\ln\left|\frac{a+x}{a-x}\right| - \frac{1}{2a}\arctan\frac{x}{a} + C$$

7. Find $\displaystyle\int \frac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3}\,dx.$

We write $\dfrac{x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4}{(x^2+2)^3} = \dfrac{Ax+B}{x^2+2} + \dfrac{Cx+D}{(x^2+2)^2} + \dfrac{Ex+F}{(x^2+2)^3}$. Then

$$x^5 - x^4 + 4x^3 - 4x^2 + 8x - 4 = (Ax+B)(x^2+2)^2 + (Cx+D)(x^2+2) + Ex + F$$
$$= Ax^5 + Bx^4 + (4A+C)x^3 + (4B+D)x^2 + (4A+2C+E)x + (4B+2D+F)$$

from which $A = 1$, $B = -1$, $C = 0$, $D = 0$, $E = 4$, $F = 0$. Thus the given integral is equal to

$$\int \frac{x-1}{x^2+2}\,dx + 4\int \frac{x\,dx}{(x^2+2)^3} = \frac{1}{2}\ln(x^2+2) - \frac{\sqrt{2}}{2}\arctan\frac{x}{\sqrt{2}} - \frac{1}{(x^2+2)^2} + C$$

8. Find $\displaystyle\int \frac{2x^2+3}{(x^2+1)^2}\,dx.$

We write $\dfrac{2x^2+3}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$. Then

$$2x^2 + 3 = (Ax+B)(x^2+1) + Cx + D = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

from which $A = 0$, $B = 2$, $A + C = 0$, $B + D = 3$. Thus $A = 0$, $B = 2$, $C = 0$, $D = 1$ and

$$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = \int \frac{2\,dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$$

For the second integral on the right, let $x = \tan z$. Then

$$\int \frac{dx}{(x^2+1)^2} = \int \frac{\sec^2 z\,dz}{\sec^4 z} = \int \cos^2 z\,dz = \frac{1}{2}z + \frac{1}{4}\sin 2z + C$$

and $$\int \frac{2x^2+3}{(x^2+1)^2}\,dx = 2\arctan x + \frac{1}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C = \frac{5}{2}\arctan x + \frac{\frac{1}{2}x}{x^2+1} + C$$

Supplementary Problems

In Problems 9 to 27, evaluate the integral at the left.

9. $\displaystyle\int \frac{dx}{x^2-9} = \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$

10. $\displaystyle\int \frac{dx}{x^2+7x+6} = \frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + C$

11. $\displaystyle\int \frac{x\,dx}{x^2-3x-4} = \frac{1}{5}\ln|(x+1)(x-4)^4| + C$

12. $\displaystyle\int \frac{x^2+3x-4}{x^2-2x-8}\,dx = x + \ln|(x+2)(x-4)^4| + C$

13. $\displaystyle\int \frac{x^2-3x-1}{x^3+x^2-2x}\,dx = \ln\left|\frac{x^{1/2}(x+2)^{3/2}}{x-1}\right| + C$

14. $\displaystyle\int \frac{x\,dx}{(x-2)^2} = \ln|x-2| - \frac{2}{x-2} + C$

15. $\displaystyle\int \frac{x^4}{(1-x)^3}\,dx = -\frac{1}{2}x^2 - 3x - \ln(1-x)^6 - \frac{4}{1-x} + \frac{1}{2(1-x)^2} + C$

16. $\displaystyle\int \frac{dx}{x^3+x} = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + C$

17. $\displaystyle\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx = \ln\sqrt{x^2+3} + \arctan x + C$

18. $\displaystyle\int \frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}\,dx = \frac{1}{2}x^2 + \ln\left|\frac{x}{\sqrt{x^2-2x+3}}\right| + C$

19. $\displaystyle\int \frac{2x^3\,dx}{(x^2+1)^2} = \ln(x^2+1) + \frac{1}{x^2+1} + C$

20. $\displaystyle\int \frac{2x^3+x^2+4}{(x^2+4)^2}\,dx = \ln(x^2+4) + \frac{1}{2}\arctan\frac{1}{2}x + \frac{4}{x^2+4} + C$

21. $\displaystyle\int \frac{x^3+x-1}{(x^2+1)^2}\,dx = \ln\sqrt{x^2+1} - \frac{1}{2}\arctan x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$

22. $\displaystyle\int \frac{x^4+8x^3-x^2+2x+1}{(x^2+x)(x^3+1)}\,dx = \ln\left|\frac{x^3-x^2+x}{(x+1)^2}\right| - \frac{3}{x+1} + \frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$

23. $\displaystyle\int \frac{x^3+x^2-5x+15}{(x^2+5)(x^2+2x+3)}\,dx = \ln\sqrt{x^2+2x+3} + \frac{5}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} - \sqrt{5}\arctan\frac{x}{\sqrt{5}} + C$

24. $\displaystyle\int \frac{x^6+7x^5+15x^4+32x^3+23x^2+25x-3}{(x^2+x+2)^2(x^2+1)^2}\,dx = \frac{1}{x^2+x+2} - \frac{3}{x^2+1} + \ln\frac{x^2+1}{x^2+x+2} + C$

25. $\displaystyle\int \frac{dx}{e^{2x}-3e^x} = \frac{1}{3e^x} + \frac{1}{9}\ln\left|\frac{e^x-3}{e^x}\right| + C$ (*Hint*: Let $e^x = u$.)

26. $\displaystyle\int \frac{\sin x\,dx}{\cos x\,(1+\cos^2 x)} = \ln\left|\frac{\sqrt{1+\cos^2 x}}{\cos x}\right| + C$ (*Hint*: Let $\cos x = u$.)

27. $\displaystyle\int \frac{(2+\tan^2\theta)\sec^2\theta\,d\theta}{1+\tan^3\theta} = \ln|1+\tan\theta| + \frac{2}{\sqrt{3}}\arctan\frac{2\tan\theta-1}{\sqrt{3}} + C$

Chapter 35

Miscellaneous Substitutions

IF AN INTEGRAND IS RATIONAL except for a radical of the form

1. $\sqrt[n]{ax+b}$, then the substitution $ax + b = z^n$ will replace it with a rational integrand.
2. $\sqrt{q+px+x^2}$, then the substitution $q + px + x^2 = (z-x)^2$ will replace it with a rational integrand.
3. $\sqrt{q+px-x^2} = \sqrt{(\alpha+x)(\beta-x)}$, then the substitution $q + px - x^2 = (\alpha+x)^2z^2$ or $q + px - x^2 = (\beta - x)^2z^2$ will replace it with a rational integrand.

(See Problems 1 to 5.)

THE SUBSTITUTION $x = 2\arctan z$ will replace any rational function of $\sin x$ and $\cos x$ with a rational function of z, since

$$\sin x = \frac{2z}{1+z^2} \qquad \cos x = \frac{1-z^2}{1+z^2} \qquad \text{and} \qquad dx = \frac{2\,dz}{1+z^2}$$

(The first and second of these relations are obtained from Fig. 35-1, and the third by differentiating $x = 2\arctan z$.) After integrating, use $z = \tan\frac{1}{2}x$ to return to the original variable. (See Problems 6 to 10.)

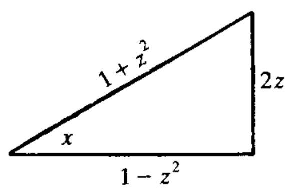

Fig. 35-1

EFFECTIVE SUBSTITUTIONS are often suggested by the form of the integrand. (See Problems 11 and 12.)

Solved Problems

1. Find $\displaystyle\int \frac{dx}{x\sqrt{1-x}}$.

Let $1 - x = z^2$. Then $x = 1 - z^2$, $dx = -2z\,dz$, and

$$\int \frac{dx}{x\sqrt{1-x}} = \int \frac{-2z\,dz}{(1-z^2)z} = -2\int \frac{dz}{1-z^2} = -\ln\left|\frac{1+z}{1-z}\right| + C = \ln\left|\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right| + C$$

2. Find $\displaystyle\int \frac{dx}{(x-2)\sqrt{x+2}}$.

Let $x+2=z^2$. Then $x=z^2-2$, $dx=2z\,dz$, and

$$\int \frac{dx}{(x-2)\sqrt{x+2}}=\int \frac{2z\,dz}{z(z^2-4)}=2\int \frac{dz}{z^2-4}=\frac{1}{2}\ln\left|\frac{z-2}{z+2}\right|+C=\frac{1}{2}\ln\left|\frac{\sqrt{x+2}-2}{\sqrt{x+2}+2}\right|+C$$

3. Find $\displaystyle\int \frac{dx}{x^{1/2}-x^{1/4}}$.

Let $x=z^4$. Then $dx=4z^3\,dz$ and

$$\int \frac{dx}{x^{1/2}-x^{1/4}}=\int \frac{4z^3\,dz}{z^2-z}=4\int \frac{z^2}{z-1}\,dz=4\int \left(z+1+\frac{1}{z-1}\right)dz$$

$$=4\left(\tfrac{1}{2}z^2+z+\ln|z-1|\right)+C=2\sqrt{x}+4\sqrt[4]{x}+\ln(\sqrt[4]{x}-1)^4+C$$

4. Find $\displaystyle\int \frac{dx}{x\sqrt{x^2+x+2}}$.

Let $x^2+x+2=(z-x)^2$. Then

$$x=\frac{z^2-2}{1+2x} \qquad dx=\frac{2(z^2+z+2)\,dz}{(1+2z)^2} \qquad \sqrt{x^2+x+2}=\frac{z^2+z+2}{1+2z}$$

and

$$\int \frac{dx}{x\sqrt{x^2+x+2}}=\int \frac{\dfrac{2(z^2+z+2)}{(1+2z)^2}}{\dfrac{z^2-2}{1+2z}\;\dfrac{z^2+z+2}{1+2z}}\,dz=2\int \frac{dz}{z^2-2}=\frac{1}{\sqrt{2}}\ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{x^2+x+2}+x-\sqrt{2}}{\sqrt{x^2+x+2}+x+\sqrt{2}}\right|+C$$

5. Find $\displaystyle\int \frac{x\,dx}{(5-4x-x^2)^{3/2}}$.

Let $5-4x-x^2=(5+x)(1-x)=(1-x)^2z^2$. Then

$$x=\frac{z^2-5}{1+z^2} \qquad dx=\frac{12z\,dz}{(1+z^2)^2} \qquad \sqrt{5-4x-x^2}=(1-x)z=\frac{6z}{1+z^2}$$

and

$$\int \frac{x\,dx}{(5-4x-x^2)^{3/2}}=\int \frac{\dfrac{z^2-5}{1+z^2}\;\dfrac{12z}{(1+z^2)^2}}{\dfrac{216z^3}{(1+z^2)^3}}\,dz=\frac{1}{18}\int\left(1-\frac{5}{z^2}\right)dz$$

$$=\frac{1}{18}\left(z+\frac{5}{z}\right)+C=\frac{5-2x}{9\sqrt{5-4x-x^2}}+C$$

In Problems 6 to 10, evaluate the integral at the left.

6. $$\int \frac{dx}{1+\sin x-\cos x}=\int \frac{\dfrac{2\,dz}{1+z^2}}{1+\dfrac{2z}{1+z^2}-\dfrac{1-z^2}{1+z^2}}=\int \frac{dz}{z(1+z)}=\ln|z|-\ln|1+z|+C$$

$$=\ln\left|\frac{z}{1+z}\right|+C=\ln\left|\frac{\tan\frac{1}{2}x}{1+\tan\frac{1}{2}x}\right|+C$$

7. $$\int \frac{dx}{3-2\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{3-2\frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{1+5z^2} = \frac{2\sqrt{5}}{5}\arctan z\sqrt{5} + C$$
$$= \frac{2\sqrt{5}}{5}\arctan(\sqrt{5}\tan\tfrac{1}{2}x) + C$$

8. $$\int \sec x\,dx = \int \frac{1+z^2}{1-z^2}\frac{2\,dz}{1+z^2} = 2\int \frac{dz}{1-z^2} = \ln\left|\frac{1+z}{1-z}\right| + C = \ln\left|\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}\right| + C$$
$$= \ln\left|\tan\left(\frac{1}{2}x + \frac{1}{4}\pi\right)\right| + C$$

9. $$\int \frac{dx}{2+\cos x} = \int \frac{\frac{2\,dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int \frac{dz}{3+z^2} = \frac{2}{\sqrt{3}}\arctan\frac{z}{\sqrt{3}} + C$$
$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\tan\frac{1}{2}x\right) + C$$

10. $$\int \frac{dx}{5+4\sin x} = \int \frac{\frac{2\,dz}{1+z^2}}{5+4\frac{2z}{1+z^2}} = \int \frac{2\,dz}{5+8z+5z^2} = \frac{2}{5}\int \frac{dz}{(z+\frac{4}{5})^2+\frac{9}{25}}$$
$$= \frac{2}{3}\arctan\frac{z+\frac{4}{5}}{\frac{3}{5}} + C = \frac{2}{3}\arctan\frac{5\tan\frac{1}{2}x+4}{3} + C$$

11. Use the substitution $1-x^3=z^2$ to find $\int x^5\sqrt{1-x^3}\,dx$.

The substitution yields $x^3 = 1-z^2$, $3x^2\,dx = -2z\,dz$, and

$$\int x^5\sqrt{1-x^3}\,dx = \int x^3\sqrt{1-x^3}\,(x^2\,dx) = \int (1-z^2)z(-\tfrac{2}{3}z\,dz) = -\frac{2}{3}\int (1-z^2)z^2\,dz$$
$$= -\frac{2}{3}\left(\frac{z^3}{3} - \frac{z^5}{5}\right) + C = -\frac{2}{45}(1-x^3)^{3/2}(2+3x^3) + C$$

12. Use $x = \dfrac{1}{z}$ to find $\int \dfrac{\sqrt{x-x^2}}{x^4}\,dx$.

The substitution yields $dx = -dz/z^2$, $\sqrt{x-x^2} = \sqrt{z-1}/z$, and

$$\int \frac{\sqrt{x-x^2}}{x^4}\,dx = \int \frac{\frac{\sqrt{z-1}}{z}\left(-\frac{dz}{z^2}\right)}{1/z^4} = -\int z\sqrt{z-1}\,dz$$

Let $z-1=s^2$. Then

$$-\int z\sqrt{z-1}\,dz = -\int (s^2+1)(s)(2s\,ds) = -2\left(\frac{s^5}{5}+\frac{s^3}{3}\right) + C$$
$$= -2\left[\frac{(z-1)^{5/2}}{5} + \frac{(z-1)^{3/2}}{3}\right] + C = -2\left[\frac{(1-x)^{5/2}}{5x^{5/2}} + \frac{(1-x)^{3/2}}{3x^{3/2}}\right] + C$$

13. Find $\int \dfrac{dx}{x^{1/2}+x^{1/3}}$.

Let $u = x^{1/6}$, so that $x = u^6$, $dx = 6u^5\,du$, $x^{1/2} = u^3$, and $x^{1/3} = u^2$. Then we obtain

$$\int \frac{6u^5\,du}{u^3 + u^2} = 6\int \frac{u^3}{u+1}\,du = 6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right) du = 6\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u+1|\right) + C$$
$$= 2x^{1/2} - 3x^{1/3} + x^{1/6} - \ln|x^{1/6} + 1| + C$$

Supplementary Problems

In Problems 14 to 39, evaluate the integral at the left.

14. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx = 2\sqrt{x} - 2\arctan\sqrt{x} + C$

15. $\displaystyle\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\ln(1+\sqrt{x}) + C$

16. $\displaystyle\int \frac{dx}{3+\sqrt{x+2}} = 2\sqrt{x+2} - 6\ln(3+\sqrt{x+2}) + C$

17. $\displaystyle\int \frac{1-\sqrt{3x+2}}{1+\sqrt{3x+2}}\,dx = -x + \frac{4}{3}\left\{\sqrt{3x+2} - \ln(1+\sqrt{3x+2})\right\} + C$

18. $\displaystyle\int \frac{dx}{\sqrt{x^2 - x + 1}} = \ln|2\sqrt{x^2 - x + 1} + 2x - 1| + C$

19. $\displaystyle\int \frac{dx}{x\sqrt{x^2 + x - 1}} = 2\arctan(\sqrt{x^2 + x - 1} + x) + C$

20. $\displaystyle\int \frac{dx}{\sqrt{6 + x - x^2}} = \arcsin\frac{2x-1}{5} + C$

21. $\displaystyle\int \frac{\sqrt{4x - x^2}}{x^3}\,dx = -\frac{(4x - x^2)^{3/2}}{6x^3} + C$

22. $\displaystyle\int \frac{dx}{(x+1)^{1/2} + (x+1)^{1/4}} = 2(x+1)^{1/2} - 4(x+1)^{1/4} + 4\ln(1 + (x+1)^{1/4}) + C$

23. $\displaystyle\int \frac{dx}{2+\sin x} = \frac{2}{\sqrt{3}}\arctan\frac{2\tan\frac{1}{2}x + 1}{\sqrt{3}} + C$

24. $\displaystyle\int \frac{dx}{1 - 2\sin x} = \frac{\sqrt{3}}{3}\ln\left|\frac{\tan\frac{1}{2}x - 2 - \sqrt{3}}{\tan\frac{1}{2}x - 2 + \sqrt{3}}\right| + C$

25. $\displaystyle\int \frac{dx}{3 + 5\sin x} = \frac{1}{4}\ln\left|\frac{3\tan\frac{1}{2}x + 1}{\tan\frac{1}{2}x + 3}\right| + C$

26. $\displaystyle\int \frac{dx}{\sin x - \cos x - 1} = \ln|\tan\tfrac{1}{2}x - 1| + C$

27. $\displaystyle\int \frac{dx}{5 + 3\sin x} = \frac{1}{2}\arctan\frac{5\tan\frac{1}{2}x + 3}{4} + C$

28. $\displaystyle\int \frac{\sin x\,dx}{1 + \sin^2 x} = \frac{\sqrt{2}}{4}\ln\left|\frac{\tan^2\frac{1}{2}x + 3 - 2\sqrt{2}}{\tan^2\frac{1}{2}x + 3 + 2\sqrt{2}}\right| + C$

29. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x} = \ln|1 + \tan\tfrac{1}{2}x| + C$

30. $\displaystyle\int \frac{dx}{2 - \cos x} = \frac{2}{\sqrt{3}}\arctan(\sqrt{3}\tan\tfrac{1}{2}x) + C$

31. $\displaystyle\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

32. $\displaystyle\int \frac{dx}{x\sqrt{3x^2 + 2x - 1}} = -\arcsin\frac{1-x}{2x} + C$ (*Hint*: Let $x = 1/z$.)

33. $\displaystyle\int \frac{(e^x - 2)e^x}{e^x + 1}\,dx = e^x - 3\ln(e^x + 1) + C$ (*Hint*: Let $e^x + 1 = z$.)

34. $\displaystyle\int \frac{\sin x \cos x}{1-\cos x}\,dx = \cos x + \ln(1-\cos x) + C$ (*Hint*: Let $\cos x = z$.)

35. $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{\sqrt{4-x^2}}{4x} + C$ (*Hint*: Let $x = 2/z$.)

36. $\displaystyle\int \frac{dx}{x^2(4+x^2)} = -\frac{1}{4x} + \frac{1}{8}\arctan\frac{2}{x} + C$

37. $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx = \tfrac{4}{5}(1+\sqrt{x})^{5/2} - \tfrac{4}{3}(1+\sqrt{x})^{3/2} + C$

38. $\displaystyle\int \frac{dx}{3(1-x^2)-(5+4x)\sqrt{1-x^2}} = \frac{2\sqrt{1+x}}{3\sqrt{1+x}-\sqrt{1-x}} + C$

39. $\displaystyle\int \frac{x^{1/2}}{x^{1/5}+1}\,dx = 10\left(\frac{1}{13}x^{13/10} - \frac{1}{11}x^{11/10} + \frac{1}{9}x^{9/10} - \frac{1}{7}x^{7/10} + \frac{1}{5}x^{5/10} - \frac{1}{3}x^{3/10} + x^{1/10} - \arctan x^{1/10}\right)$

$+ C$ (*Hint*: Let $u = x^{1/10}$.)

Chapter 36

Integration of Hyperbolic Functions

INTEGRATION FORMULAS. The following formulas are direct consequences of the differentiation formulas of Chapter 20.

28. $\int \sinh x\, dx = \cosh x + C$

29. $\int \cosh x\, dx = \sinh x + C$

30. $\int \tanh x\, dx = \ln \cosh x + C$

31. $\int \coth x\, dx = \ln |\sinh x| + C$

32. $\int \operatorname{sech}^2 x\, dx = \tanh x + C$

33. $\int \operatorname{csch}^2 x\, dx = -\coth x + C$

34. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$

35. $\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

36. $\int \dfrac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \dfrac{x}{a} + C$

37. $\int \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \dfrac{x}{a} + C, \quad x > a > 0$

38. $\int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{a} \tanh^{-1} \dfrac{x}{a} + C, \quad x^2 < a^2$

39. $\int \dfrac{dx}{x^2 - a^2} = -\dfrac{1}{a} \coth^{-1} \dfrac{x}{a} + C, \quad x^2 > a^2$

Solved Problems

In Problems 1 to 13, evaluate the integral at the left.

1. $\int \sinh \tfrac{1}{2}x\, dx = 2 \int \sinh \tfrac{1}{2}x\, d(\tfrac{1}{2}x) = 2 \cosh \tfrac{1}{2}x + C$

2. $\int \cosh 2x\, dx = \tfrac{1}{2} \int \cosh 2x\, d(2x) = \tfrac{1}{2} \sinh 2x + C$

3. $\int \operatorname{sech}^2 (2x - 1)\, dx = \tfrac{1}{2} \int \operatorname{sech}^2 (2x - 1)\, d(2x - 1) = \tfrac{1}{2} \tanh (2x - 1) + C$

4. $\int \operatorname{csch} 3x \coth 3x\, dx = \tfrac{1}{3} \int \operatorname{csch} 3x \coth 3x\, d(3x) = -\tfrac{1}{3} \operatorname{csch} 3x + C$

5. $\int \operatorname{sech} x\, dx = \int \dfrac{1}{\cosh x}\, dx = \int \dfrac{\cosh x}{\cosh^2 x} = \int \dfrac{\cosh x}{1 + \sinh^2 x}\, dx = \arctan (\sinh x) + C$

6. $\int \sinh^2 x\, dx = \tfrac{1}{2} \int (\cosh 2x - 1)\, dx = \tfrac{1}{4} \sinh 2x - \tfrac{1}{2}x + C$

7. $\int \tanh^2 2x\, dx = \int (1 - \operatorname{sech}^2 2x)\, dx = x - \tfrac{1}{2} \tanh 2x + C$

8. $\int \cosh^3 \frac{1}{2}x\,dx = \int (1+\sinh^2 \frac{1}{2}x)\cosh \frac{1}{2}x\,dx = 2\sinh \frac{1}{2}x + \frac{2}{3}\sinh^3 \frac{1}{2}x + C$

9. $\int \operatorname{sech}^4 x\,dx = \int (1-\tanh^2 x)\operatorname{sech}^2 x\,dx = \tanh x - \frac{1}{3}\tanh^3 x + C$

10. $\int e^x \cosh x\,dx = \int e^x \dfrac{e^x+e^{-x}}{2}\,dx = \dfrac{1}{2}\int (e^{2x}+1)\,dx = \dfrac{1}{4}e^{2x} + \dfrac{1}{2}x + C$

11. $\int x\sinh x\,dx = \int x\dfrac{e^x-e^{-x}}{2}\,dx = \dfrac{1}{2}\int xe^x\,dx - \dfrac{1}{2}\int xe^{-x}\,dx$

$$= \frac{1}{2}(xe^x - e^x) - \frac{1}{2}(-xe^{-x} - e^{-x}) + C = x\frac{e^x+e^{-x}}{2} - \frac{e^x-e^{-x}}{2} + C$$
$$= x\cosh x - \sinh x + C$$

12. $\int \dfrac{dx}{\sqrt{4x^2-9}} = \dfrac{1}{2}\cosh^{-1}\dfrac{2x}{3} + C$ **13.** $\int \dfrac{dx}{9x^2-25} = -\dfrac{1}{15}\coth^{-1}\dfrac{3x}{5} + C$

14. Find $\int \sqrt{x^2+4}\,dx$.

Let $x = 2\sinh z$. Then $dx = 2\cosh z\,dz$, $\sqrt{x^2+4} = 2\cosh z$, and

$$\int \sqrt{x^2+4}\,dx = 4\int \cosh^2 z\,dz = 2\int (\cosh 2z + 1)\,dz = \sinh 2z + 2z + C$$
$$= 2\sinh z\cosh z + 2z + C = \tfrac{1}{2}x\sqrt{x^2+4} + 2\sinh^{-1}\tfrac{1}{2}x + C$$

15. Find $\int \dfrac{dx}{x\sqrt{1-x^2}}$.

Let $x = \operatorname{sech} z$. Then $dx = -\operatorname{sech} z\tanh z\,dz$, $1-x^2 = \tanh z$, and

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\int \frac{\operatorname{sech} z\tanh z}{\operatorname{sech} z\tanh z}\,dz = -\int dz = -z + C = -\operatorname{sech}^{-1} x + C$$

Supplementary Problems

In Problems 16 to 39, evaluate the integral at the left.

16. $\int \sinh 3x\,dx = \frac{1}{3}\cosh 3x + C$

17. $\int \cosh \frac{1}{4}x\,dx = 4\sinh \frac{1}{4}x + C$

18. $\int \coth \frac{3}{2}x\,dx = \frac{2}{3}\ln|\sinh \frac{3}{2}x| + C$

19. $\int \operatorname{csch}^2(1+3x)\,dx = -\frac{1}{3}\coth(1+3x) + C$

20. $\int \operatorname{sech} 2x\tanh 2x\,dx = -\frac{1}{2}\operatorname{sech} 2x + C$

21. $\int \operatorname{csch} x\,dx = \ln\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}} + C$

22. $\int \cosh^2 \frac{1}{2}x\,dx = \frac{1}{2}(\sinh x + x) + C$

23. $\int \coth^2 3x\,dx = x - \frac{1}{3}\coth 3x + C$

24. $\int \sinh^3 x\,dx = \frac{1}{3}\cosh^3 x - \cosh x + C$

25. $\int e^x \sinh x\,dx = \frac{1}{4}e^{2x} - \frac{1}{2}x + C$

26. $\int e^{2x}\cosh x\,dx = \frac{1}{6}e^{3x} + \frac{1}{2}e^{x} + C$

27. $\int x\cosh x\,dx = x\sinh x - \cosh x + C$

28. $\int x^2\sinh x\,dx = (x^2+2)\cosh x - 2x\sinh x + C$

29. $\int \sinh^3 x\cosh^2 x\,dx = \frac{1}{5}\cosh^5 x - \frac{1}{3}\cosh^3 x + C$

30. $\int \sinh x \ln\cosh^2 x\,dx = \cosh x\,(\ln\cosh^2 x - 2) + C$

31. $\int \frac{dx}{\sqrt{x^2+9}} = \sinh^{-1}\frac{x}{3} + C$

32. $\int \frac{dx}{\sqrt{x^2-25}} = \cosh^{-1}\frac{x}{5} + C$

33. $\int \frac{dx}{4-9x^2} = \frac{1}{6}\tanh^{-1}\frac{3}{2}x + C$

34. $\int \frac{dx}{16x^2-9} = -\frac{1}{12}\coth^{-1}\frac{4}{3}x + C$

35. $\int \sqrt{x^2-9}\,dx = \frac{x}{2}\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}\frac{x}{3} + C$

36. $\int \frac{dx}{\sqrt{x^2-2x+17}} = \sinh^{-1}\frac{x-1}{4} + C$

37. $\int \frac{dx}{4x^2+12x+5} = -\frac{1}{4}\coth^{-1}\left(x+\frac{3}{2}\right) + C$

38. $\int \frac{x^2}{(x^2+4)^{3/2}}\,dx = \sinh^{-1}\frac{x}{2} - \frac{x}{\sqrt{x^2+4}} + C$

39. $\int \frac{\sqrt{x^2+1}}{x^2}\,dx = \sinh^{-1} x - \frac{\sqrt{1+x^2}}{x} + C$

Chapter 37

Applications of Indefinite Integrals

WHEN THE EQUATION $y = f(x)$ of a curve is known, the slope m at any point $P(x, y)$ on it is given by $m = f'(x)$. Conversely, when the slope of a curve at a point $P(x, y)$ on it is given by $m = dy/dx = f'(x)$, a family of curves, $y = f(x) + C$, may be found by integration. To single out a particular curve of the family, it is necessary to assign or to determine a particular value of C. This may be done by prescribing that the curve pass through a given point. (See Problems 1 to 4.)

AN EQUATION $s = f(t)$, where s is the distance at time t of a body from a fixed point in its (straight-line) path, completely defines the motion of the body. The velocity and acceleration at time t are given by

$$v = \frac{ds}{dt} = f'(t) \qquad \text{and} \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$$

Conversely, if the velocity (or acceleration) is known at time t, together with the position (or position and velocity) at some given instant, usually at $t = 0$, the equation of motion may be obtained. (See Problems 7 to 10.)

Solved Problems

1. Find the equation of the family of curves whose slope at any point is equal to the negative of twice the abscissa of the point. Find the curve of the family which passes through the point $(1, 1)$.

We are given that $dy/dx = -2x$. Then $dy = -2x\,dx$, from which $\int dy = \int -2x\,dx$, and $y = -x^2 + C$. This is the equation of a family of parabolas.

Setting $x = 1$ and $y = 1$ in the equation of the family yields $1 = -1 + C$ or $C = 2$. The equation of the curve passing through the point $(1, 1)$ is then $y = -x^2 + 2$.

2. Find the equation of the family of curves whose slope at any point $P(x, y)$ is $m = 3x^2y$. Find the equation of the curve of the family which passes through the point $(0, 8)$.

Since $m = \dfrac{dy}{dx} = 3x^2y$, we have $\dfrac{dy}{y} = 3x^2\,dx$. Then $\ln y = x^3 + C = x^3 + \ln c$ and $y = ce^{x^3}$. When $x = 0$ and $y = 8$, then $8 = ce^0 = c$. The equation of the required curve is $y = 8e^{x^3}$.

3. At every point of a certain curve, $y'' = x^2 - 1$. Find the equation of the curve if it passes through the point $(1, 1)$ and is there tangent to the line $x + 12y = 13$.

Here $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(y') = x^2 - 1$. Then $\int \dfrac{d}{dx}(y')\,dx = \int (x^2 - 1)\,dx$ and $y' = \dfrac{x^3}{3} - x + C_1$.

At $(1, 1)$, the slope y' of the curve equals the slope $-\frac{1}{12}$ of the line. Then $-\frac{1}{12} = \frac{1}{3} - 1 + C_1$, from which $C_1 = \frac{7}{12}$. Hence $y' = dy/dx = \frac{1}{3}x^3 - x + \frac{7}{12}$, and integration yields

$$\int dy = \int \left(\tfrac{1}{3}x^3 - x + \tfrac{7}{12}\right) dx \qquad \text{or} \qquad y = \tfrac{1}{12}x^4 - \tfrac{1}{2}x^2 + \tfrac{7}{12}x + C_2$$

At $(1, 1)$, $1 = \frac{1}{12} - \frac{1}{2} + \frac{7}{12} + C_2$ and $C_2 = \frac{5}{6}$. The required equation is $y = \frac{1}{12}x^4 - \frac{1}{2}x^2 + \frac{7}{12}x + \frac{5}{6}$.

4. The family of *orthogonal trajectories* of a given system of curves is another system of curves, each of which cuts every curve of the given system at right angles. Find the equations of the orthogonal trajectories of the family of hyperbolas $x^2 - y^2 = c$.

At any point $P(x, y)$, the slope of the hyperbola through the point is given by $m_1 = x/y$, and the slope of the orthogonal trajectory through P is given by $m_2 = dy/dx = -y/x$. Then

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \qquad \text{so that} \qquad \ln|y| = -\ln|x| + \ln C' \qquad \text{or} \qquad |xy| = C'$$

The required equation is $xy = \pm C'$ or, simply, $xy = C$.

5. A certain quantity q increases at a rate proportional to itself. If $q = 25$ when $t = 0$ and $q = 75$ when $t = 2$, find q when $t = 6$.

Since $dq/dt = kq$, we have $dq/q = k\,dt$. Integration yields $\ln q = kt + \ln c$ or $q = ce^{kt}$.
When $t = 0$, $q = 25 = ce^0$; hence, $c = 25$ and $q = 25e^{kt}$.
When $t = 2$, $q = 25e^{2k} = 75$; then $e^{2k} = 3 = e^{1.10}$. So $k = 0.55$ and $q = 25e^{0.55t}$.
Finally, when $t = 6$, $q = 25e^{0.55t} = 25e^{3.3} = 25(e^{1.1})^3 = 25(27) = 675$.

6. A substance is being transformed into another at a rate proportional to the untransformed amount. If the original amount is 50 and is 25 when $t = 3$, when will $\frac{1}{10}$ of the substance remain untransformed?

Let q represent the amount transformed in time t. Then $dq/dt = k(50 - q)$, from which

$$\frac{dq}{50 - q} = k\,dt \qquad \text{so that} \qquad \ln(50 - q) = -kt + \ln c \qquad \text{or} \qquad 50 - q = ce^{-kt}$$

When $t = 0$, $q = 0$ and $c = 50$; thus $50 - q = 50e^{-kt}$.
When $t = 3$, $50 - q = 25 = 50e^{-3k}$; then $e^{-3k} = 0.5 = e^{-0.69}$, $k = 0.23$, and $50 - q = 50 - e^{-0.23t}$. When the untransformed amount is 5, $50e^{-0.23t} = 5$; then $e^{-0.23t} = 0.1 = e^{-2.30}$ and $t = 10$.

7. A ball is rolled over a level lawn with initial velocity 25 ft/sec. Due to friction, the velocity decreases at the rate of 6 ft/sec^2. How far will the ball roll?

Here $dv/dt = -6$. So $v = -6t + C_1$. When $t = 0$, $v = 25$; hence $C_1 = 25$ and $v = -6t + 25$.
Since $v = ds/dt = -6t + 25$, integration yields $s = -3t^2 + 25t + C_2$. When $t = 0$, $s = 0$; hence $C_2 = 0$ and $s = -3t^2 + 25t$.
When $v = 0$, $t = \frac{25}{6}$; hence, the ball rolls for $\frac{25}{6}$ sec before coming to rest. In that time it rolls a distance $s = -3(\frac{25}{6})^2 + 25(\frac{25}{6}) = -\frac{625}{12} + \frac{625}{6} = \frac{625}{12}$ ft.

8. A stone is thrown straight down from a stationary balloon, 10,000 ft above the ground, with a speed of 48 ft/sec. Locate the stone and find its speed 20 sec later.

Take the upward direction as positive. When the stone leaves the balloon, it has acceleration $a = dv/dt = -32$ ft/sec^2 and velocity $v = -32t + C_1$.
When $t = 0$, $v = -48$; hence $C_1 = -48$. Then $v = ds/dt = -32t - 48$ and $s = -16t^2 - 48t + C_2$.
When $t = 0$, $s = 10{,}000$; hence $C_2 = 10{,}000$ and $s = -16t^2 - 48t + 10{,}000$.
When $t = 20$,

$$s = -16(20)^2 - 48(20) + 10{,}000 = 2640 \qquad \text{and} \qquad v = -32(20) - 48 = -688$$

After 20 sec, the stone is 2640 ft above the ground and its speed is 688 ft/sec.

9. A ball is dropped from a balloon that is 640 ft above the ground and rising at the rate of 48 ft/sec. Find (*a*) the greatest distance above the ground attained by the ball, (*b*) the time the ball is in the air, and (*c*) the speed of the ball when it strikes the ground.

Take the upward direction as positive. Then $a = dv/dt = -32$ ft/sec^2 and $v = -32t + C_1$.

When $t = 0$, $v = 48$; hence $C_1 = 48$. Then $v = ds/dt = -32t + 48$ and $s = -16t^2 + 48t + C_2$. When $t = 0$, $s = 640$; hence $C_2 = 640$ and $s = -16t^2 + 48t + 640$.

(*a*) When $v = 0$, $t = \frac{3}{2}$ and $s = -16(\frac{3}{2})^2 + 48(\frac{3}{2}) + 640 = 676$. The greatest height attained by the ball is 676 ft.

(*b*) When $s = 0$, $-16t^2 + 48t + 640 = 0$ and $t = -5, 8$. The ball is in the air for 8 sec.

(*c*) When $t = 8$, $v = -32(8) + 48 = -208$. The ball strikes the ground with speed 208 ft/sec.

10. The velocity with which water will flow from a small orifice in a tank, at a depth h ft below the surface, is $0.6\sqrt{2gh}$ ft/sec, where $g = 32$ ft/sec^2. Find the time required to empty an upright cylindrical tank of height 5 ft and radius 1 ft through a round 1-in hole in the bottom.

Let h be the depth of the water at time t. The water flowing out in time dt generates a cylinder of height $v\,dt$ ft, radius 1/24 ft, and volume $\pi(1/24)^2\, v\, dt = 0.6\pi(1/24)^2\sqrt{2gh}\, dt$ ft^3.

Let $-dh$ represent the corresponding drop in the surface level. The loss in volume is $-\pi(1)^2\, dh$ ft^3. Then $0.6\pi(1/24)^2(8\sqrt{h}\, dt) = -\pi\, dh$, or $dt = -(120\, dh)/\sqrt{h}$ and $t = -240\sqrt{h} + C$.

At $t = 0$, $h = 5$ and $C = 240\sqrt{5}$; thus $t = -240\sqrt{h} + 240\sqrt{5}$.

When the tank is empty, $h = 0$ and $t = 240\sqrt{5}$ sec $= 9$ min, approximately.

Supplementary Problems

11. Find the equation of the family of curves having the given slope, and the equation of the curve of the family which passes through the given point, in each of the following:

(*a*) $m = 4x$; $(1, 5)$ (*b*) $m = \sqrt{x}$; $(9, 18)$ (*c*) $m = (x-1)^3$; $(3, 0)$
(*d*) $m = 1/x^2$; $(1, 2)$ (*e*) $m = x/y$; $(4, 2)$ (*f*) $m = x^2/y^3$; $(3, 2)$
(*g*) $m = 2y/x$; $(2, 8)$ (*h*) $m = xy/(1+x^2)$; $(3, 5)$

Ans. (*a*) $y = 2x^2 + C$, $y = 2x^2 + 3$; (*b*) $3y = 2x^{3/2} + C$, $3y = 2x^{3/2}$; (*c*) $4y = (x-1)^4 + C$, $4y = (x-1)^4 - 16$; (*d*) $xy = Cx - 1$, $xy = 3x - 1$; (*e*) $x^2 - y^2 = C$, $x^2 - y^2 = 12$; (*f*) $3y^4 = 4x^3 + C$, $3y^4 = 4x^3 - 60$; (*g*) $y = Cx^2$, $y = 2x^2$; (*h*) $y^2 = C(1+x^2)$, $2y^2 = 5(1+x^2)$

12. (*a*) For a certain curve, $y'' = 2$. Find its equation given that it passes through $P(2, 6)$ with slope 10. *Ans.* $y = x^2 + 6x - 10$

(*b*) For a certain curve, $y'' = 6x - 8$. Find its equation given that it passes through $P(1, 0)$ with slope 4. *Ans.* $y = x^3 - 4x^2 + 9x - 6$

13. A particle moves along a straight line from the origin (at $t = 0$) with the given velocity v. Find the distance the particle moves during the interval between the two given times t.

(*a*) $v = 4t + 1$; 0, 4 (*b*) $v = 6t + 3$; 1, 3 (*c*) $v = 3t^2 + 2t$; 2, 4
(*d*) $v = \sqrt{t} + 5$; 4, 9 (*e*) $v = 2t - 2$; 0, 5 (*f*) $v = t^2 - 3t + 2$; 0, 4

Ans. (*a*) 36; (*b*) 30; (*c*) 68; (*d*) $37\frac{2}{3}$; (*e*) 17; (*f*) $5\frac{2}{3}$

14. Find the equation of the family of orthogonal trajectories of the system of parabolas $y^2 = 2x + C$.
Ans. $y = Ce^{-x}$

15. A particle moves in a straight line from the origin (at $t = 0$) with given initial velocity v_0 and acceleration a. Find s at time t.
(*a*) $a = 32$, $v_0 = 2$ (*b*) $a = -32$; $v_0 = 96$ (*c*) $a = 12t^2 + 6t$; $v_0 = -3$ (*d*) $a = 1/\sqrt{t}$; $v_0 = 4$

Ans. (*a*) $s = 16t^2 + 2t$; (*b*) $s = -16t^2 + 96t$; (*c*) $s = t^4 + t^3 - 3t$; (*d*) $s = \frac{4}{3}(t^{3/2} + 3t)$

16. A car is slowing down at the rate 0.8 ft/sec^2. How far will the car move before it stops if its speed is initially 15 mi/hr? *Ans.* $302\frac{1}{2}$ ft

17. A particle is projected vertically upward from a point 112 ft above the ground with initial velocity 96 ft/sec. (*a*) How fast is it moving when it is 240 ft above the ground? (*b*) When will it reach the highest point in its path? (*c*) At what speed will it strike the ground?

Ans. (*a*) 32 ft/sec; (*b*) after 3 sec; (*c*) 128 ft/sec

18. A block of ice slides down a chute with acceleration 4 ft/sec^2. The chute is 60 ft long, and the ice reaches the bottom in 5 sec. What are the initial velocity of the ice and the velocity when it is 20 ft from the bottom of the chute? *Ans.* 2 ft/sec; 18 ft/sec

19. What constant acceleration is required (*a*) to move a particle 50 ft in 5 sec; (*b*) to slow a particle from a velocity of 45 ft/sec to a dead stop in 15 ft? *Ans.* (*a*) 4 ft/sec^2; (*b*) $-67\frac{1}{2}$ ft/sec^2

20. The bacteria in a certain culture increase according to $dN/dt = 0.25N$. If originally $N = 200$, find N when $t = 8$. *Ans.* 1478

Chapter 38

The Definite Integral

THE DEFINITE INTEGRAL. Let $a \le x \le b$ be an interval on which a given function $f(x)$ is continuous. Divide the interval into n subintervals $h_1, h_2, \ldots, h_n$ by the insertion of $n-1$ points $\xi_1, \xi_2, \ldots, \xi_{n-1}$, where $a < \xi_1 < \xi_2 < \cdots < \xi_{n-1} < b$, and relabel a as ξ_0 and b as ξ_n. Denote the length of the subinterval h_1 by $\Delta_1 x = \xi_1 - \xi_0$, of h_2 by $\Delta_2 x = \xi_2 - \xi_1, \ldots$, of h_n by $\Delta_n x = \xi_n - \xi_{n-1}$. (This is done in Fig. 38-1. The lengths are directed distances, each being positive in view of the above inequality.) On each subinterval select a point (x_1 on the subinterval h_1, x_2 on $h_2, \ldots, x_n$ on h_n) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = f(x_1)\,\Delta_1 x + f(x_2)\,\Delta_2 x + \cdots + f(x_n)\,\Delta_n x \tag{38.1}$$

each term being the product of the length of a subinterval and the value of the function at the selected point on that subinterval. Denote by λ_n the length of the longest subinterval appearing in (*38.1*). Now let the number of subintervals increase indefinitely in such a manner that $\lambda_n \to 0$. (One way of doing this would be to bisect each of the original subintervals, then bisect each of these, and so on.) Then

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x \tag{38.2}$$

exists and is the same for all methods of subdividing the interval $a \le x \le b$, so long as the condition $\lambda_n \to 0$ is met, and for all choices of the points x_k in the resulting subintervals.

Fig. 38-1

A proof of this theorem is beyond the scope of this book. In Problems 1 to 3 the limit is evaluated for selected functions $f(x)$. It must be understood, however, that for an arbitrary function this procedure is too difficult to attempt. Moreover, to succeed in the evaluations made here, it is necessary to prescribe some relation among the lengths of the subintervals (we take them all of equal length) and to follow some pattern in choosing a point on each subinterval (for example, choose the left-hand endpoint or the right-hand endpoint or the midpoint of each subinterval).

By agreement, we write

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$$

The symbol $\int_a^b f(x)\,dx$ is read "the *definite integral* of $f(x)$, with respect to x, from $x = a$ to $x = b$." The function $f(x)$ is called the *integrand*; a and b are called, respectively, the *lower* and *upper limits* (boundaries) *of integration*. (See Problems 1 to 3.)

We have defined $\int_a^b f(x)\,dx$ when $a < b$. The other cases are taken care of by the following definitions:

$$\int_a^a f(x)\,dx = 0 \tag{38.3}$$

$$\text{If } a < b, \text{ then } \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \tag{38.4}$$

PROPERTIES OF DEFINITE INTEGRALS. If $f(x)$ and $g(x)$ are continuous on the interval of integration $a \le x \le b$, then

Property 38.1: $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, for any constant c

(For a proof, see Problem 4.)

Property 38.2: $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Property 38.3: $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, for $a < c < b$

Property 38.4 (first mean-value theorem): $\int_a^b f(x)\,dx = (b-a)f(x_0)$ for at least one value $x = x_0$ between a and b.

(For a proof, see Problem 5.)

Property 38.5: If $F(u) = \int_a^u f(x)\,dx$, then $\frac{d}{du}F(u) = f(u)$

(For a proof, see Problem 6.)

FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS. If $f(x)$ is continuous on the interval $a \le x \le b$, and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

(For a proof, see Problem 7.)

EXAMPLE 1: (*a*) Take $f(x) = c$, a constant, and $F(x) = cx$; then $\int_a^b c\,dx = cx\Big|_a^b = c(b-a)$.
(*b*) Take $f(x) = x$ and $F(x) = \frac{1}{2}x^2$; then $\int_0^5 x\,dx = \frac{1}{2}x^2\Big|_0^5 = \frac{25}{2} - 0 = \frac{25}{2}$.
(*c*) Take $f(x) = x^3$ and $F(x) = \frac{1}{4}x^4$; then $\int_1^3 x^3\,dx = \frac{1}{4}x^4\Big|_1^3 = \frac{81}{4} - \frac{1}{4} = 20$.
These results should be compared with those of Problems 1 to 3. The reader can show that *any* indefinite integral of $f(x)$ may be used by redoing (*c*) with $F(x) = \frac{1}{4}x^4 + C$.

(See Problems 8 to 20.)

THE THEOREM OF BLISS. If $f(x)$ and $g(x)$ are continuous on the interval $a \le x \le b$, if the interval is divided into subintervals as before, and if two points are selected in each subinterval (that is, x_k and X_k in the kth subinterval), then

$$\lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)g(X_k)\,\Delta_k x = \int_a^b f(x)g(x)\,dx$$

We note first that the theorem is true if the points x_k and X_k are identical. The force of the theorem is that when the points of each pair are distinct, the result is the same as if they were coincident. An intuitive feeling for the validity of the theorem follows from writing

$$\sum_{k=1}^{n} f(x_k)g(X_k)\,\Delta_k x = \sum_{k=1}^{n} f(x_k)g(x_k)\,\Delta_k x + \sum_{k=1}^{n} f(x_k)[g(X_k) - g(x_k)]\,\Delta_k x$$

and noting that as $n \to +\infty$ (that is, as $\Delta_k x \to 0$) x_k and X_k must become more nearly identical and, since $g(x)$ is continuous, $g(X_k) - g(x_k)$ must then go to zero.

In evaluating definite integrals directly from the definition, we sometimes make use of the following summation formulas:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{38.5}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{38.6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \tag{38.7}$$

These formulas can be proved by mathematical induction.

Solved Problems

In Problems 1 to 3, evaluate the integral by setting up S_n and obtaining the limit as $n \to +\infty$.

1. $\int_a^b c\,dx = c(b-a)$, c constant

Let the interval $a \le x \le b$ be divided into n equal subintervals of length $\Delta x = (b-a)/n$. Since the integrand is $f(x) = c$, then $f(x_k) = c$ for any choice of the point x_k on the kth subinterval, and

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = \sum_{k=1}^{n} c(\Delta x) = (c + c + \cdots + c)(\Delta x) = nc\,\Delta x = nc\,\frac{b-a}{n} = c(b-a)$$

Hence
$$\int_a^b c\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} c(b-a) = c(b-a)$$

2. $\int_0^5 x\,dx = \frac{25}{2}$

Let the interval $0 \le x \le 5$ be divided into n equal subintervals of length $\Delta x = 5/n$. Take the points x_k as the right-hand endpoints of the subintervals; that is, $x_1 = \Delta x$, $x_2 = 2\,\Delta x, \ldots, x_n = n\,\Delta x$, as shown in Fig. 38-2. Then

$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x = \sum_{k=1}^{n} (k\,\Delta x)\,\Delta x = (1 + 2 + \cdots + n)(\Delta x)^2 = \frac{n(n+1)}{2}\left(\frac{5}{n}\right)^2 = \frac{25}{2}\left(1 + \frac{1}{n}\right)$$

and
$$\int_0^5 x\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{25}{2}$$

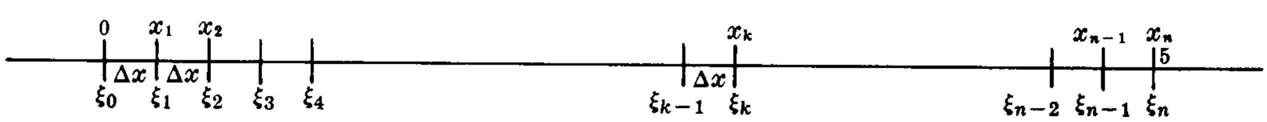

Fig. 38-2

3. $\displaystyle\int_1^3 x^3\,dx = 20$

Let the interval $1 \le x \le 3$ be divided into n subintervals of length $\Delta x = 2/n$.

First method: Take the points x_k as the left-hand endpoints of the subintervals, as in Fig. 38-3; that is, $x_1 = 1$, $x_2 = 1 + \Delta x, \ldots, x_n = 1 + (n-1)\,\Delta x$. Then

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} f(x_k)\,\Delta_k n = x_1^3\,\Delta x + x_2^3\,\Delta x + \cdots + x_n^3\,\Delta x \\
&= \{1 + (1+\Delta x)^3 + (1 + 2\,\Delta x)^3 + \cdots + [1 + (n-1)\,\Delta x]^3\}\,\Delta x \\
&= \{n + 3[1 + 2 + \cdots + (n-1)]\,\Delta x + 3[1^2 + 2^2 + \cdots + (n-1)^2](\Delta x)^2 \\
&\qquad + [1^3 + 2^3 + \cdots + (n-1)^3](\Delta x)^3\}\,\Delta x \\
&= \left[n + 3\,\frac{(n-1)n}{1\cdot 2}\,\frac{2}{n} + 3\,\frac{(n-1)n(2n-1)}{1\cdot 2\cdot 3}\left(\frac{2}{n}\right)^2 + \frac{(n-1)^2n^2}{(1\cdot 2)^2}\left(\frac{2}{n}\right)^3\right]\frac{2}{n} \\
&= 2 + \left(6 - \frac{6}{n}\right) + \left(8 - \frac{12}{n} + \frac{4}{n^2}\right) + \left(4 - \frac{8}{n} + \frac{4}{n^2}\right) = 20 - \frac{26}{n} + \frac{8}{n^2}
\end{aligned}$$

and
$$\int_1^3 x^3\,dx = \lim_{n\to+\infty}\left(20 - \frac{26}{n} + \frac{8}{n^2}\right) = 20$$

Fig. 38-3

Second method: Take the points x_k as the midpoints of the subintervals, as in Fig. 38-4; that is, $x_1 = 1 + \frac{1}{2}\,\Delta x$, $x_2 = 1 + \frac{3}{2}\,\Delta x, \ldots, x_n = 1 + \dfrac{2n-1}{2}\,\Delta x$. Then

$$\begin{aligned}
S_n &= \left[\left(1 + \frac{1}{2}\,\Delta x\right)^3 + \left(1 + \frac{3}{2}\,\Delta x\right)^3 + \cdots + \left(1 + \frac{2n-1}{2}\,\Delta x\right)^3\right]\Delta x \\
&= \left\{\left[1 + 3\left(\frac{1}{2}\right)\Delta x + 3\left(\frac{1}{2}\right)^2(\Delta x)^2 + \left(\frac{1}{2}\right)^3(\Delta x)^3\right] + \left[1 + 3\left(\frac{3}{2}\right)(\Delta x) + 3\left(\frac{3}{2}\right)^2(\Delta x)^2 + \left(\frac{3}{2}\right)^3\left(\Delta x\right)^3\right] + \cdots \right. \\
&\qquad \left. + \left[1 + 3\,\frac{2n-1}{2}\,\Delta x + 3\left(\frac{2n-1}{2}\right)^2(\Delta x)^2 + \left(\frac{2n-1}{2}\right)^3(\Delta x)^3\right]\right\}\Delta x \\
&= n\,\frac{2}{n} + \frac{3}{2}\,n^2\left(\frac{2}{n}\right)^2 + \frac{1}{4}\,(4n^3 - n)\left(\frac{2}{n}\right)^3 + \frac{1}{8}\,(2n^4 - n^2)\left(\frac{2}{n}\right)^4 \\
&= 2 + 6 + \left(8 - \frac{2}{n^2}\right) + \left(4 - \frac{2}{n^2}\right) = 20 - \frac{4}{n^2}
\end{aligned}$$

and
$$\int_1^3 x^3\,dx = \lim_{n\to+\infty}\left(20 - \frac{4}{n^2}\right) = 20$$

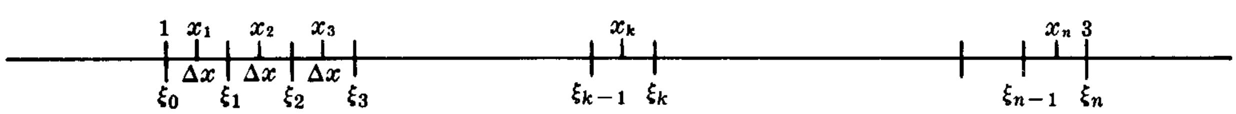

Fig. 38-4

4. Prove: $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.

For a proper subdivision of the interval $a \le x \le b$ and any choice of points on the subintervals,

$$S_n = \sum_{k=1}^{n} cf(x_k)\,\Delta_k x = c \sum_{k=1}^{n} f(x_k)\,\Delta_k x$$

Then
$$\int_a^b cf(x)\,dx = c \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x = c\int_a^b f(x)\,dx$$

5. Prove the first mean-value theorem of the integral calculus: If $f(x)$ is continuous on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx = (b-a)f(x_0)$ for at least one value $x = x_0$ between a and b.

The theorem is true, by Example 1(*a*), when $f(x) = c$, a constant. Otherwise, let m be the absolute minimum value, and M be the absolute maximum value, of $f(x)$ on the interval $a \le x \le b$. For any proper subdivision of the interval and any choice of the points x_k on the subintervals,

$$\sum_{k=1}^{n} m\,\Delta_k x < \sum_{k=1}^{n} f(x_k)\,\Delta_k x < \sum_{k=1}^{n} M\,\Delta_k x$$

Now when $n \to +\infty$, we have

$$\int_a^b m\,dx < \int_a^b f(x)\,dx < \int_a^b M\,dx$$

which, by Problem 1, becomes

$$m(b-a) < \int_a^b f(x)\,dx < M(b-a)$$

Then
$$m < \frac{1}{b-a}\int_a^b f(x)\,dx < M$$

so that $\frac{1}{b-a}\int_a^b f(x)\,dx = N$, where N is some number between m and M. Now since $f(x)$ is continuous on the interval $a \le x \le b$, it must, by Property 8.1, take on at least once every value from m to M. Hence, there must be a value of x, say $x = x_0$, such that $f(x_0) = N$. Then

$$\frac{1}{b-a}\int_a^b f(x)\,dx = N = f(x_0) \qquad \text{and} \qquad \int_a^b f(x)\,dx = (b-a)f(x_0)$$

6. Prove: If $F(u) = \int_a^u f(x)\,dx$, then $\frac{d}{du}F(u) = f(u)$.

We have
$$F(u+\Delta u) - F(u) = \int_a^{u+\Delta u} f(x)\,dx - \int_a^u f(x)\,dx$$

By Properties 38.3 and 38.4, this becomes

$$F(u+\Delta u) - F(u) = \int_u^a f(x)\,dx + \int_a^{u+\Delta u} f(x)\,dx = \int_u^{u+\Delta u} f(x)\,dx = f(u_0)\,\Delta u$$

where $u < u_0 < u + \Delta u$. Then

$$\frac{F(u+\Delta u) - F(u)}{\Delta u} = f(u_0) \qquad \text{and} \qquad \frac{dF}{du} = \lim_{\Delta u\to 0} \frac{F(u+\Delta u)-F(u)}{\Delta u} = \lim_{\Delta u \to 0} f(u_0) = f(u)$$

since $u_0 \to u$ as $\Delta u \to 0$.

This property is most frequently stated as:

$$\text{If } F(x) = \int_a^x f(x)\,dx, \text{ then } F'(x) = f(x)\,. \tag{1}$$

The use of the letter u above was merely an attempt to avoid the possibility of confusing the roles of the several x's. Note carefully in (*1*) that $F(x)$ is a function of the upper limit x of integration and not of the *dummy* letter x in $f(x)\,dx$. In other words, the property might also be stated as:

$$\text{If } F(x) = \int_a^x f(t)\,dt, \text{ then } F'(x) = f(x)\,.$$

It follows from (*1*) that $F(x)$ is simply an indefinite integral of $f(x)$.

7. Prove: If $f(x)$ is continuous on the interval $a \le x \le b$, and if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Use the last statement in Problem 6 to write $\int_a^x f(x)\,dx = F(x) + C$. When the upper limit of integration is $x = a$, we have

$$\int_a^a f(x)\,dx = 0 = F(a) + C \qquad \text{so} \qquad C = -F(a)$$

Then $\int_a^x f(x)\,dx = F(x) - F(a)$, and when the upper limit of integration is $x = b$, we have, as required, $\int_a^b f(x)\,dx = F(b) - F(a)$.

In Problems 8 to 17, use the fundamental theorem of integral calculus to evaluate the integral at the left.

8. $\displaystyle\int_{-1}^{1} (2x^2 - x^3)\,dx = \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_{-1}^{1} = \left(\frac{2}{3} - \frac{1}{4}\right) - \left(-\frac{2}{3} - \frac{1}{4}\right) = \frac{4}{3}$

9. $\displaystyle\int_{-3}^{-1} \left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = \left[-\frac{1}{x} + \frac{1}{2x^2}\right]_{-3}^{-1} = \left(1 + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{1}{18}\right) = \frac{10}{9}$

10. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_1^4 = 2(\sqrt{4} - \sqrt{1}) = 2$

11. $\displaystyle\int_{-2}^{3} e^{-x/2}\,dx = [-2e^{-x/2}]_{-2}^{3} = -2(e^{-3/2} - e) = 4.9904$

12. $\displaystyle\int_{-6}^{-10} \frac{dx}{x+2} = [\ln|x+2|]_{-6}^{-10} = \ln 8 - \ln 4 = \ln 2$

13. $\displaystyle\int_{\pi/2}^{3\pi/4} \sin x\,dx = [-\cos x]_{\pi/2}^{3\pi/4} = -(-\tfrac{1}{2}\sqrt{2} - 0) = \tfrac{1}{2}\sqrt{2}$

14. $\displaystyle\int_{-2}^{2} \frac{dx}{x^2+4} = \left[\frac{1}{2}\arctan\frac{1}{2}x\right]_{-2}^{2} = \frac{1}{2}\left[\frac{1}{4}\pi - \left(-\frac{1}{4}\pi\right)\right] = \frac{1}{4}\pi$

15. $\displaystyle\int_{-5}^{-3} \sqrt{x^2-4}\,dx = \left[\frac{1}{2}x\sqrt{x^2-4} - 2\ln|x + \sqrt{x^2-4}|\right]_{-5}^{-3} = \frac{5}{2}\sqrt{21} - \frac{3}{2}\sqrt{5} - 2\ln\frac{3-\sqrt{5}}{5-\sqrt{21}}$

16. $\displaystyle\int_{-1}^{2} \frac{dx}{x^2-9} = \left[\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|\right]_{-1}^{2} = \frac{1}{6}\left(\ln\frac{1}{5} - \ln 2\right) = \frac{1}{6}\ln 0.1$

17. $\displaystyle\int_1^e \ln x\,dx = [x\ln x - x]_1^e = (e\ln e - e) - (\ln 1 - 1) = 1$

18. Find $\displaystyle\int_3^6 xy\,dx$ when $x = 6\cos\theta$, $y = 2\sin\theta$.

We shall express x, y, and dx in the integral in terms of the parameter θ and $d\theta$, change the limits of integration to corresponding values of the parameter, and evaluate the resulting integral. We have, immediately, $dx = -6\sin\theta\, d\theta$. Also, when $x = 6\cos\theta = 6$, then $\theta = 0$; and when $x = 6\cos\theta = 3$, then $\theta = \pi/3$. Hence

$$\int_3^6 xy\,dx = \int_{\pi/3}^0 (6\cos\theta)(2\sin\theta)(-6\sin\theta)\,d\theta = -72\int_{\pi/3}^0 \sin^2\theta\cos\theta\,d\theta$$
$$= [-24\sin^3\theta]_{\pi/3}^0 = -24[0 - (\sqrt{3}/2)^3] = 9\sqrt{3}$$

19. Find $\displaystyle\int_0^{2\pi/3} \frac{d\theta}{5 + 4\cos\theta}$.

The substitution $\theta = 2\arctan z$ (Fig. 38-5) yields $\displaystyle\int \frac{d\theta}{5+4\cos\theta} = \int \frac{\dfrac{2\,dz}{1+z^2}}{5 + 4\dfrac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{9+z^2}$. To determine the z limits of integration, note that when $\theta = 0$, $z = 0$; when $\theta = 2\pi/3$, $\arctan z = \pi/3$ and $z = \sqrt{3}$. Then

$$\int_0^{2\pi/3} \frac{d\theta}{5+4\cos\theta} = 2\int_0^{\sqrt{3}} \frac{dz}{9+z^2} = \frac{2}{3}\left[\arctan\frac{z}{3}\right]_0^{\sqrt{3}} = \frac{\pi}{9}$$

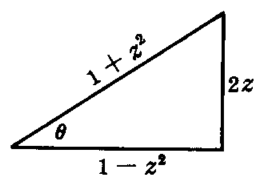

Fig. 38-5

20. Find $\displaystyle\int_0^{\pi/3} \frac{dx}{1 - \sin x}$.

The substitution $x = 2\arctan z$ yields $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{\dfrac{2\,dz}{1+z^2}}{1 - \dfrac{2z}{1+z^2}} = \int \frac{2\,dz}{(1-z)^2}$. When $x = 0$, $\arctan z = 0$ and $z = 0$; when $x = \pi/3$, $\arctan z = \pi/6$ and $z = \sqrt{3}/3$. Then

$$\int_0^{\pi/3} \frac{dx}{1-\sin x} = 2\int_0^{\sqrt{3}/3} \frac{dz}{(1-z)^2} = \left[\frac{2}{1-z}\right]_0^{\sqrt{3}/3} = \frac{2}{1-\sqrt{3}/3} - 2 = \sqrt{3}+1$$

Supplementary Problems

21. Evaluate $\displaystyle\int_a^b c\,dx$ of Problem 1 by dividing the interval $a \le x \le b$ into n subintervals of lengths $\Delta_1 x$, $\Delta_2 x, \ldots, \Delta_n x$. Note that $\displaystyle\sum_{k=1}^{n} \Delta_k x = b - a$.

22. Evaluate $\displaystyle\int_0^5 x\,dx$ of Problem 2 using subintervals of equal length and (*a*) choosing the points x_k as the left-hand endpoints of the subintervals; (*b*) choosing the points x_k as the midpoints of the subintervals; and (*c*) choosing the points x_k one-third of the way into each subinterval, that is, taking $x_1 = \frac{1}{3}\Delta x$, $x_2 = \frac{4}{3}\Delta x, \ldots$.

23. Evaluate $\int_1^4 x^2\,dx = 21$ using subintervals of equal length and choosing the points x_k as (*a*) the right-hand endpoints of the subintervals; (*b*) the left-hand endpoints of the subintervals; (*c*) the midpoints of the subintervals.

24. Using the same choice of subintervals and points as in Problem 23(*a*), evaluate $\int_1^4 x\,dx$ and $\int_1^4 (x^2+x)\,dx$, and verify that $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

25. Evaluate $\int_1^2 x^2\,dx$ and $\int_2^4 x^2\,dx$. Compare the sum with the result of Problem 23 to verify that

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx \quad \text{for } a<c<b$$

26. Evaluate $\int_0^1 e^x\,dx = e-1$. $\left(\textit{Hint}:\ S_n = \sum_{k=1}^{n} e^{k\,\Delta x}\,\Delta x = e^{\Delta x}(e-1)\dfrac{\Delta x}{e^{\Delta x}-1},\ \text{and}\ \lim_{n\to+\infty}\dfrac{\Delta x}{e^{\Delta x}-1} = \lim_{\Delta x\to 0}\dfrac{\Delta x}{e^{\Delta x}-1}\ \text{is indeterminate of the type } 0/0.\right)$

27. Prove Properties 38.2 and 38.3.

28. Use the fundamental theorem to evaluate each integral:

(*a*) $\int_0^2 (2+x)\,dx = 6$

(*b*) $\int_0^2 (2-x)^2\,dx = \frac{8}{3}$

(*c*) $\int_0^3 (3-2x+x^2)\,dx = 9$

(*d*) $\int_{-1}^2 (1-t^2)t\,dt = -\frac{9}{4}$

(*e*) $\int_1^4 (1-u)\sqrt{u}\,du = -\frac{116}{15}$

(*f*) $\int_1^8 \sqrt{1+3x}\,dx = 26$

(*g*) $\int_0^2 x^2(x^3+1)\,dx = \frac{40}{3}$

(*h*) $\int_0^3 \dfrac{dx}{\sqrt{1+x}} = 2$

(*i*) $\int_0^1 x(1-\sqrt{x})^2\,dx = \frac{1}{30}$

(*j*) $\int_4^8 \dfrac{x\,dx}{\sqrt{x^2-15}} = 6$

(*k*) $\int_0^a \sqrt{a^2-x^2}\,dx = \frac{1}{4}a^2\pi$

(*l*) $\int_{-1}^1 x^2\sqrt{4-x^2}\,dx = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}$

(*m*) $\int_3^4 \dfrac{dx}{25-x^2} = \dfrac{1}{5}\ln\dfrac{3}{2}$

(*n*) $\int_{-1/2}^0 \dfrac{x^3\,dx}{x^2+x+1} = \dfrac{\sqrt{3}\pi}{9} - \dfrac{5}{8}$

(*o*) $\int_2^4 \dfrac{\sqrt{16-x^2}}{x}\,dx = 4\ln(2+\sqrt{3}) - 2\sqrt{3}$

(*p*) $\int_8^{27} \dfrac{dx}{x-x^{1/3}} = \dfrac{3}{2}\ln\dfrac{8}{3}$

(*q*) $\int_0^1 \ln(x^2+1)\,dx = \ln 2 + \frac{1}{2}\pi - 2$

(*r*) $\int_0^{2\pi} \sin\frac{1}{2}t\,dt = 4$

(*s*) $\int_0^{\pi/3} x^2\sin 3x\,dx = \frac{1}{27}(\pi^2-4)$

(*t*) $\int_0^{\pi/2} \dfrac{dx}{3+\cos 2x} = \dfrac{\sqrt{2}\pi}{8}$

29. Show that $\int_3^5 \dfrac{dx}{\sqrt{x^2+16}} = \int_{-5}^{-3} \dfrac{dx}{\sqrt{x^2+16}}$.

30. Evaluate $\int_{\theta=0}^{\theta=2\pi} y\,dx = 3\pi$, given $x = \theta - \sin\theta$, $y = 1-\cos\theta$.

31. Evaluate $\int_1^4 \sqrt{1+(y')^2}\,dx = \frac{15}{2} + \frac{1}{4}\ln 2$, given $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$.

32. Evaluate $\int_2^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{2}e^2(e-1)$, given $x = e^t \cos t$, $y = e^t \sin t$.

33. Use the appropriate reduction formulas (Chapter 31) to establish Wallis' formulas:

$$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdots(n-3)(n-1)}{2\cdot 4\cdots(n-2)n}\,\dfrac{\pi}{2} & \text{if } n \text{ is even and } >0 \\[2ex] \dfrac{2\cdot 4\cdots(n-3)(n-1)}{1\cdot 3\cdots(n-2)n} & \text{if } n \text{ is odd and } >1 \end{cases}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{1\cdot 3\cdots(m-1)\cdot 1\cdot 3\cdots(n-1)}{2\cdot 4\cdots(m+n-2)(m+n)}\,\dfrac{\pi}{2} & \text{if } m \text{ and } n \text{ are even and } >0 \\[2ex] \dfrac{2\cdot 4\cdots(m-3)(m-1)}{(n+1)(n+3)\cdots(n+m)} & \text{if } m \text{ is odd and } >1 \\[2ex] \dfrac{2\cdot 4\cdots(n-3)(n-1)}{(m+1)(m+3)\cdots(m+n)} & \text{if } n \text{ is odd and } >1 \end{cases}$$

34. Evaluate each integral:

(a) $\int_3^{11} \sqrt{2x+3}\, dx = \frac{98}{3}$ (b) $\int_0^{\pi/4} \frac{\cos 2x - 1}{\cos 2x + 1}\, dx = \frac{1}{4}\pi - 1$

(c) $\int_4^9 \frac{1-\sqrt{x}}{1+\sqrt{x}}\, dx = 4\ln\frac{3}{4} - 1$ (d) $\int_0^{\sqrt{2}} x^3 e^{x^2}\, dx = \frac{1}{2}(e^2+1)$

(e) $\int_{\pi/4}^{3\pi/4} \frac{\sin x\, dx}{\cos^2 x - 5\cos x + 4} = \frac{1}{3}\ln\frac{7+3\sqrt{2}}{7-3\sqrt{2}}$

(f) $\int_{-2}^{-1} \frac{x-1}{\sqrt{x^2-4x+3}}\, dx = \ln\frac{3-2\sqrt{2}}{4-\sqrt{15}} + 2\sqrt{2} - \sqrt{15}$

(g) $\int_{\pi/6}^{\pi/3} \frac{dx}{\sin 2x} = \ln\sqrt{3}$

(h) $\int_1^3 \ln(x + \sqrt{x^2-1})\, dx = 3\ln(3+2\sqrt{2}) - 2\sqrt{2}$

(i) $\int_{-1}^{-2} \frac{dx}{\sqrt{x^2+2x+2}} = \ln(\sqrt{2}-1)$ (j) $\int_{1/4}^{3/4} \frac{(x+1)\, dx}{x^2(x-1)} = 4\ln\frac{1}{3} - \frac{8}{3}$

(k) $\int_{-8}^{-3} \frac{(x+2)\, dx}{x(x-2)^2} = \frac{1}{2}\ln\frac{3}{4} + \frac{1}{5}$ (l) $\int_0^{\pi/4} \frac{dx}{2+\tan x} = \frac{1}{5}\ln\frac{3\sqrt{2}}{4} + \frac{\pi}{10}$

35. Prove (*38.5*) to (*38.7*).

36. Prove: $\frac{d}{dx}\int_x^b f(u)\, du = -f(x)$.

37. Prove: $\frac{d}{dx}\int_{h(x)}^{g(x)} f(u)\, du = f(g(x))g'(x) - f(h(x))h'(x)$.

38. Evaluate $\frac{d}{dx}\int_1^x \sin u\, du = \sin x$.

39. Evaluate $\frac{d}{dx}\int_x^0 u^2\, du = -x^2$.

40. Evaluate $\frac{d}{dx}\int_0^{\sin x} u^3\, du = \sin^3 x \cos x$.

41. Evaluate $\frac{d}{dx}\int_{x^2}^{4x} \cos u\, du = 4\cos 4x - 2x\cos x^2$.

Chapter 39

Plane Areas by Integration

AREA AS THE LIMIT OF A SUM. If $f(x)$ is continuous and nonnegative on the interval $a \le x \le b$, the definite integral $\int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k)\,\Delta_k x$ can be given a geometric interpretation. Let the interval $a \le x \le b$ be subdivided and points x_k be selected as in the preceding chapter. Through each of the endpoints $\xi_0 = a$, ξ_1, $\xi_2, \ldots, \xi_n = b$ erect perpendiculars to the x axis, thus dividing into n strips the portion of the plane bounded above by the curve $y = f(x)$, below by the x axis, and laterally by the abscissas $x = a$ and $x = b$. Approximate each strip as a rectangle whose base is the lower base of the strip and whose altitude is the ordinate erected at the point x_k of the subinterval. The area of the kth approximating rectangle, shown in Fig. 39-1, is $f(x_k)\,\Delta_k x$. Hence $\sum_{k=1}^{n} f(x_k)\,\Delta_k x$ is simply the sum of the areas of the n approximating rectangles.

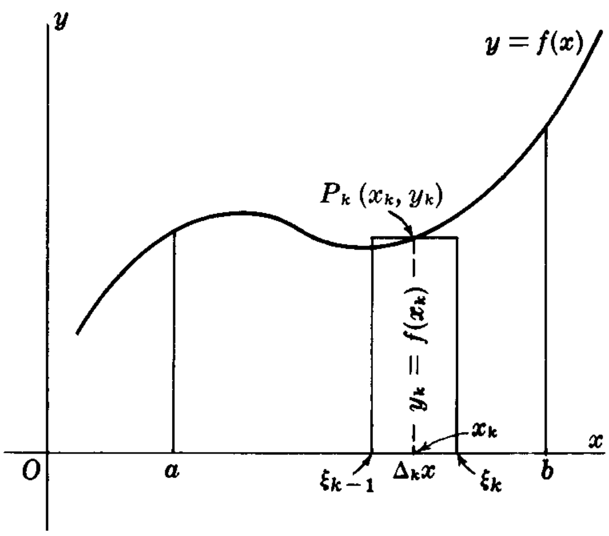

Fig. 39-1

The limit of this sum, as the number of strips is indefinitely increased in the manner prescribed in Chapter 38, is $\int_a^b f(x)\,dx$; it is also, by definition, the area of the portion of the plane described above, or, more briefly, the area under the curve from $x = a$ to $x = b$. (See Problems 1 and 2.)

Similarly, if $x = g(y)$ is continuous and nonnegative on the interval $c \le y \le d$, the definite integral $\int_c^d g(y)\,dy$ is by definition the area bounded by the curve $x = g(y)$, the y axis, and the ordinates $y = c$ and $y = d$. (See Problem 3.)

If $y = f(x)$ is continuous and nonpositive on the interval $a \le x \le b$, then $\int_a^b f(x)\,dx$ is negative, indicating that the area lies below the x axis. Similarly, if $x = g(y)$ is continuous and nonpositive on the interval $c \le y \le d$, then $\int_c^d g(y)\,dy$ is negative, indicating that the area lies to the left of the y axis. (See Problem 4.)

If $y = f(x)$ changes sign on the interval $a \le x \le b$, or if $x = g(y)$ changes sign on the interval $c \le y \le d$, then the area "under the curve" is given by the sum of two or more definite integrals. (See Problem 5.)

AREAS BY INTEGRATION. The steps in setting up a definite integral that yields a required area are:

1. Make a sketch showing the area sought, a representative (kth) strip, and the approximating rectangle. We shall generally show the representative subinterval of length Δx (or Δy), with the point x_k (or y_k) on this subinterval as its midpoint.
2. Write the area of the approximating rectangle and the sum for the n rectangles.
3. Assume the number of rectangles to increase indefinitely, and apply the fundamental theorem of the preceding chapter.

(See Problems 6 to 14.)

AREAS BETWEEN CURVES. Assume that $f(x)$ and $g(x)$ are continuous functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$. Then the area A of the region R between the graphs of $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ (see Fig. 39-2) is given by

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx \qquad (39.1)$$

That is, the area A is the difference between the area $\int_a^b f(x)\,dx$ of the region above the x axis and below $y = f(x)$ and the area $\int_a^b g(x)\,dx$ of the region above the x axis and below $y = g(x)$.

Formula (*39.1*) holds when one or both of the curves $y = f(x)$ and $y = g(x)$ lie partially or completely below the x axis, that is, when we assume only that $g(x) \le f(x)$ for $a \le x \le b$, as in Fig. 39-3.

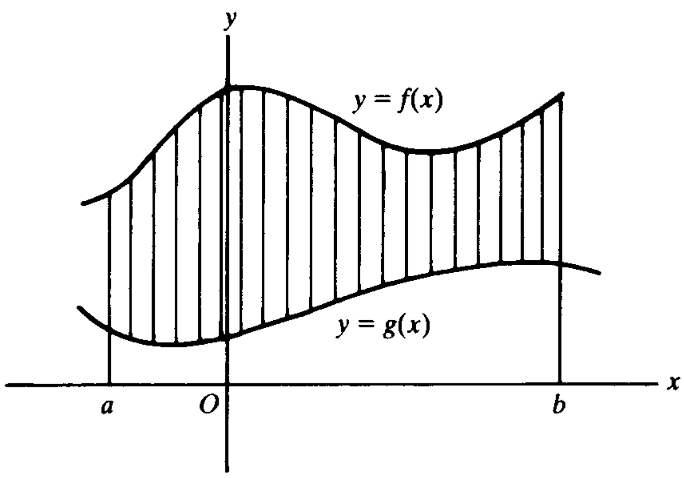

Fig. 39-2

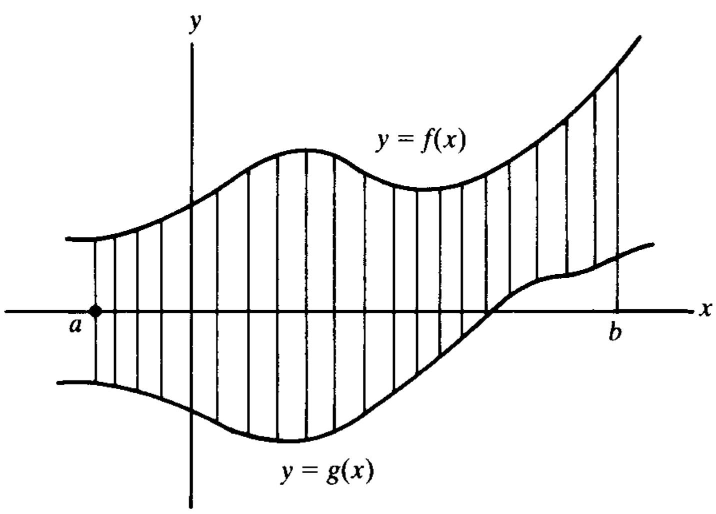

Fig. 39-3

Solved Problems

1. Find the area bounded by the curve $y = x^2$, the x axis, and the ordinates $x = 1$ and $x = 3$.

Figure 39-4 shows the area $KLMN$ sought, a representative strip $RSTU$, and its approximating rectangle $RVWU$. For this rectangle, the base is $\Delta_k x$, the altitude is $y_k = f(x_k) = x_k^2$, and the area is $x_k^2\, \Delta_k x$. Then

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} x_k^2\, \Delta_k x = \int_1^3 x^2\, dx = \left[\frac{x^3}{3}\right]_1^3 = 9 - \frac{1}{3} = \frac{26}{3} \text{ square units}$$

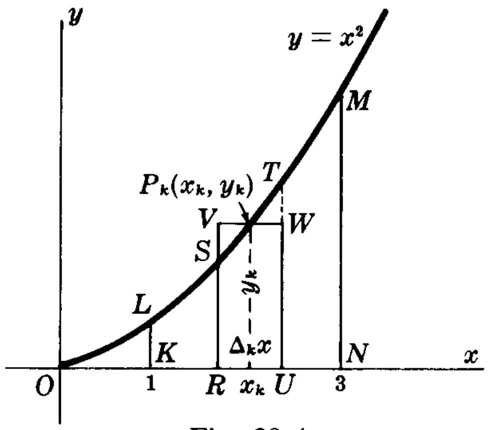

Fig. 39-4

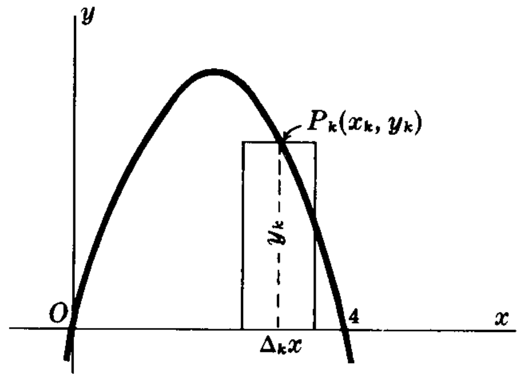

Fig. 39-5

2. Find the area lying above the x axis and under the parabola $y = 4x - x^2$.

The given curve crosses the x axis at $x = 0$ and $x = 4$. When vertical strips are used, these values become the limits of integration. For the approximating rectangle shown in Fig. 39-5, the width is $\Delta_k x$, the height is $y_k = 4x_k - x_k^2$, and the area is $(4x_k - x_k^2)\,\Delta_k x$. Then

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} (4x_k - x_k^2)\,\Delta_k x = \int_0^4 (4x - x^2)\, dx = [2x^2 - \tfrac{1}{3}x^3]_0^4 = \tfrac{32}{3} \text{ square units}$$

With the complete procedure, as given above, always in mind, an abbreviation of the work is possible. It will be seen that, aside from the limits of integration, the definite integral can be formulated once the area of the approximating rectangle has been set down.

3. Find the area bounded by the parabola $x = 8 + 2y - y^2$, the y axis, and the lines $y = -1$ and $y = 3$.

Here we slice the area into horizontal strips. For the approximating rectangle shown in Fig. 39-6, the width is Δy, the length is $x = 8 + 2y - y^2$, and the area is $(8 + 2y - y^2)\,\Delta y$. The required area is

$$A = \int_{-1}^{3} (8 + 2y - y^2)\, dy = \left[8y + y^2 - \frac{y^3}{3}\right]_{-1}^{3} = \frac{92}{3} \text{ square units}$$

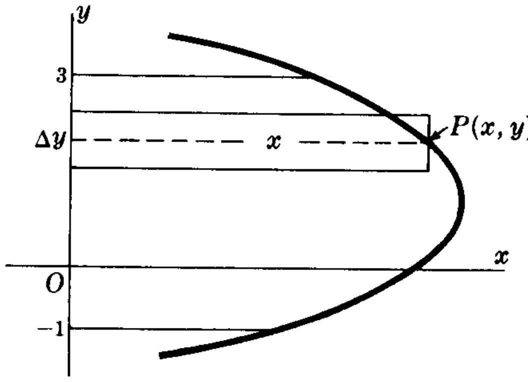

Fig. 39-6

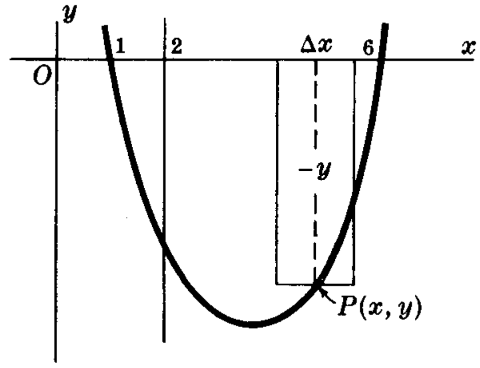

Fig. 39-7

4. Find the area bounded by the parabola $y = x^2 - 7x + 6$, the x axis, and the lines $x = 2$ and $x = 6$.

For the approximating rectangle shown in Fig. 39-7, the width is Δx, the height is $-y = -(x^2 - 7x + 6)$, and the area is $-(x^2 - 7x + 6)\,\Delta x$. The required area is then

$$A = \int_2^6 -(x^2 - 7x + 6)\,dx = -\left(\frac{x^3}{3} - \frac{7x^2}{2} + 6x\right)\Big]_2^6 = \frac{56}{3} \text{ square units}$$

5. Find the area between the curve $y = x^3 - 6x^2 + 8x$ and the x axis.

The curve crosses the x axis at $x = 0$, $x = 2$, and $x = 4$, as shown in Fig. 39-8. For vertical strips, the area of the approximating rectangle with base on the interval $0 < x < 2$ is $(x^3 - 6x^2 + 8x)\,\Delta x$, and the area of the portion lying above the x axis is given by $\int_0^2 (x^3 - 6x^2 - 8x)\,dx$. The area of the approximating rectangle with base on the interval $2 < x < 4$ is $-(x^3 - 6x^2 + 8x)\,\Delta x$, and the area of the portion lying below the x axis is given by $\int_2^4 -(x^3 - 6x^2 + 8x)\,dx$. The required area is, therefore,

$$A = \int_0^2 (x^3 - 6x^2 + 8x)\,dx + \int_2^4 -(x^3 - 6x^2 + 8x)\,dx = \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_0^2 - \left[\frac{x^4}{4} - 2x^3 + 4x^2\right]_2^4$$
$$= 4 + 4 = 8 \text{ square units}$$

The use of two definite integrals is necessary here, since the integrand changes sign on the interval of integration. Failure to note this would have resulted in the incorrect integral $\int_0^4 (x^3 - 6x^2 + 8x)\,dx = 0$.

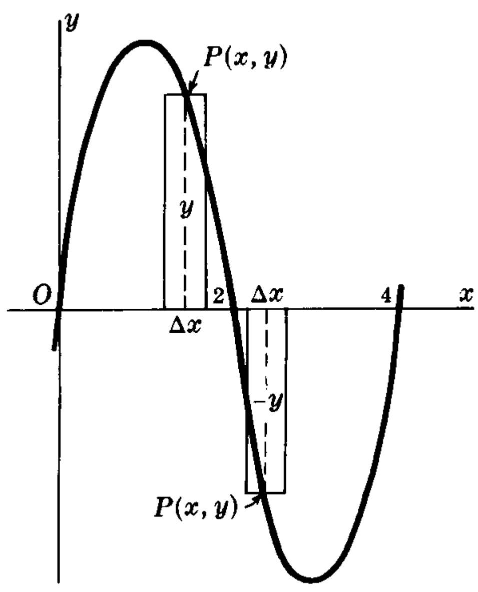

Fig. 39-8

6. Find the area bounded by the parabola $x = 4 - y^2$ and the y axis.

The parabola crosses the x axis at the point $(4, 0)$, and the y axis at the points $(0, 2)$ and $(0, -2)$. We shall give two solutions.

Using horizontal strips: For the approximating rectangle of Fig. 39-9(a), the width is Δy, the length is $4 - y^2$, and the area is $(4 - y^2)\,\Delta y$. The limits of integration of the resulting definite integral are $y = -2$ and $y = 2$. However, the area lying below the x axis is equal to that lying above. Hence, we have, for the required area,

$$A = \int_{-2}^2 (4 - y^2)\,dy = 2\int_0^2 (4 - y^2)\,dy = 2\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{32}{3} \text{ square units}$$

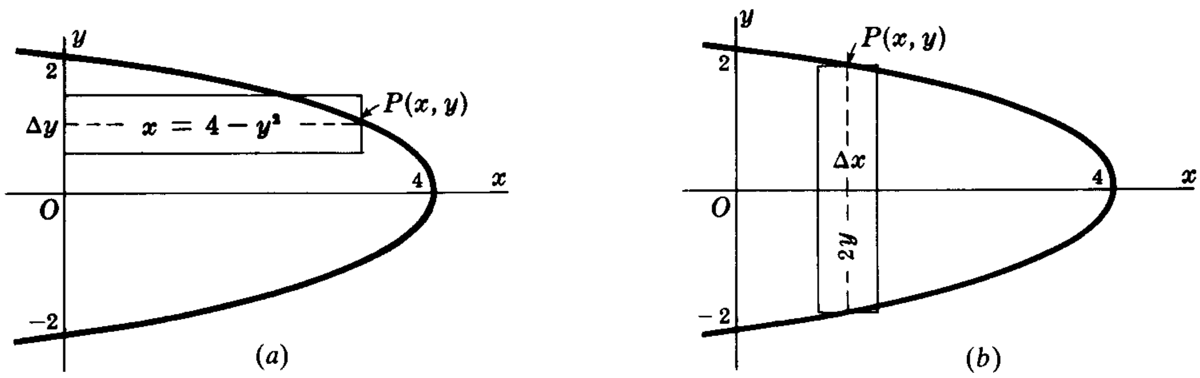

Fig. 39-9

Using vertical strips: For the approximating rectangle of Fig. 39-9(*b*), the width is Δx, the height is $2y = 2\sqrt{4-x}$, and the area is $2\sqrt{4-x}\,\Delta x$. The limits of integration are $x = 0$ and $x = 4$. Hence the required area is

$$\int_0^4 2\sqrt{4-x}\,dx = [-\tfrac{4}{3}(4-x)^{3/2}]_0^4 = \tfrac{32}{3} \text{ square units}$$

7. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.

The line intersects the parabola at the points $(1, -2)$ and $(4, 4)$. Fig. 39-10 shows clearly that when vertical strips are used, certain strips run from the line to the parabola, and others from one branch of the parabola to the other branch; however, when horizontal strips are used, each strip runs from the parabola to the line. We give both solutions here to show the superiority of one over the other and to indicate that both methods should be considered before beginning to set up a definite integral.

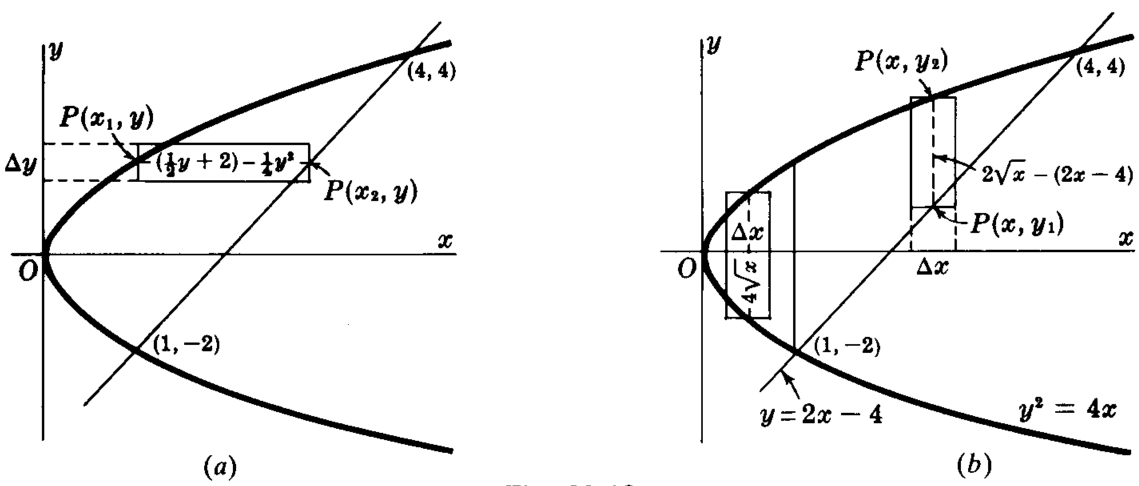

Fig. 39-10

Using horizontal strips (Fig. 39-10(*a*)): For the approximating rectangle of Fig. 39-10(*a*), the width is Δy, the length is [(value of x of the line) − (value of x of the parabola)] $= (\tfrac{1}{2}y + 2) - \tfrac{1}{4}y^2 = 2 + \tfrac{1}{2}y - \tfrac{1}{4}y^2$, and the area is $(2 + \tfrac{1}{2}y - \tfrac{1}{4}y^2)\,\Delta y$. The required area is

$$A = \int_{-2}^{4} (2 + \tfrac{1}{2}y - \tfrac{1}{4}y^2)\,dy = \left[2y + \frac{y^2}{4} - \frac{y^3}{12}\right]_{-2}^{4} = 9 \text{ square units}$$

Using vertical strips (Fig. 39-10(*b*)): Divide the area A into two parts with the line $x = 1$. For the approximating rectangle to the left of this line, the width is Δx, the height (making use of symmetry) is $2y = 4\sqrt{x}$, and the area is $4\sqrt{x}\,\Delta x$. For the approximating rectangle to the right, the width is Δx, the height is $2\sqrt{x} - (2x - 4) = 2\sqrt{x} - 2x + 4$, and the area is $(2\sqrt{x} - 2x + 4)\,\Delta x$. The required area is

$$A = \int_0^1 4\sqrt{x}\,dx + \int_1^4 (2\sqrt{x} - 2x + 4)\,dx = [\tfrac{8}{3}x^{3/2}]_0^1 + [\tfrac{4}{3}x^{3/2} - x^2 + 4x]_1^4$$

$$= \tfrac{8}{3} + \tfrac{19}{3} = 9 \text{ square units}$$

8. Find the area bounded by the parabolas $y = 6x - x^2$ and $y = x^2 - 2x$.

The parabolas intersect at the points $(0, 0)$ and $(4, 8)$. It is readily seen in Fig. 39-11 that vertical slicing will yield the simpler solution.

For the approximating rectangle, the width is Δx, the height is [(value of y of the upper boundary) − (value of y of the lower boundary)] $= (6x - x^2) - (x^2 - 2x) = 8x - 2x^2$, and the area is $(8x - 2x^2)\,\Delta x$. The required area is

$$A = \int_0^4 (8x - 2x^2)\,dx = [4x^2 - \tfrac{2}{3}x^3]_0^4 = \tfrac{64}{3} \text{ square units}$$

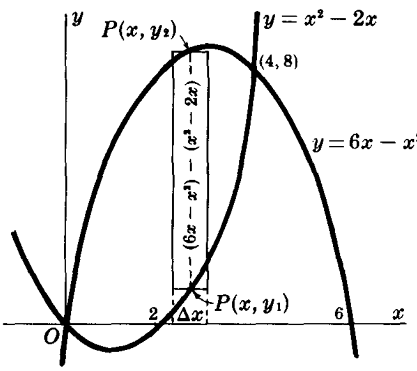

Fig. 39-11

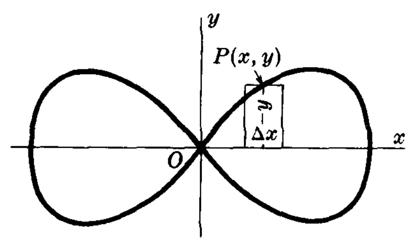

Fig. 39-12

9. Find the area enclosed by the curve $y^2 = x^2 - x^4$.

The curve is symmetric with respect to the coordinate axes. Hence the required area is four times the portion lying in the first quadrant.

For the approximating rectangle shown in Fig. 39-12, the width is Δx, the height is $y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$, and the area is $x\sqrt{1 - x^2}\,\Delta x$. Hence the required area is

$$A = 4\int_0^1 x\sqrt{1 - x^2}\,dx = [-\tfrac{4}{3}(1 - x^2)^{3/2}]_0^1 = \tfrac{4}{3} \text{ square units}$$

10. Find the smaller area cut from the circle $x^2 + y^2 = 25$ by the line $x = 3$.

Based on Fig. 39-13,

$$A = \int_3^5 2y\,dx = 2\int_3^5 \sqrt{25 - x^2}\,dx = 2\left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\arcsin\frac{x}{5}\right]_3^5$$
$$= \left(\frac{25}{2}\pi - 12 - 25\arcsin\frac{3}{5}\right) \text{ square units}$$

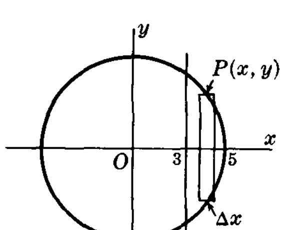

Fig. 39-13

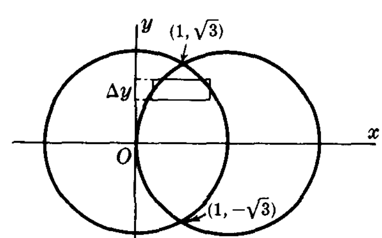

Fig. 39-14

11. Find the area common to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 4x$.

The circles intersect in the points $(1, \pm\sqrt{3})$. The approximating rectangle shown in Fig. 39-14 extends from $x = 2 - \sqrt{4 - y^2}$ to $x = \sqrt{4 - y^2}$. Then

$$A = 2\int_0^{\sqrt{3}} [\sqrt{4-y^2} - (2 - \sqrt{4-y^2})]\, dy = 4\int_0^{\sqrt{3}} (\sqrt{4-y^2} - 1)\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{4-y^2} + 2\arcsin\frac{1}{2}y - y\right]_0^{\sqrt{3}} = \left(\frac{8\pi}{3} - 2\sqrt{3}\right) \text{ square units}$$

12. Find the area of the loop of the curve $y^2 = x^4(4 + x)$. (See Fig. 39-15.)

From the figure, $A = \int_{-4}^{0} 2y\, dx = 2\int_{-4}^{0} x^2\sqrt{4+x}\, dx$. Let $4 + x = z^2$; then

$$A = 4\int_0^2 (z^2 - 4)^2 z^2\, dz = 4\left[\frac{z^7}{7} - \frac{8z^5}{5} + \frac{16z^3}{3}\right]_0^2 = \frac{4096}{105} \text{ square units}$$

13. Find the area of an arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

A single arch is described as θ varies from 0 to 2π (see Fig. 39-16). Then $dx = (1 - \cos\theta)\, d\theta$ and

$$A = \int_{\theta=0}^{\theta=2\pi} y\, dx = \int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)\, d\theta = \int_0^{2\pi} \left(\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} = 3\pi \text{ square units}$$

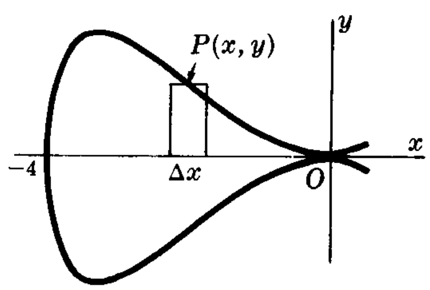

Fig. 39-15

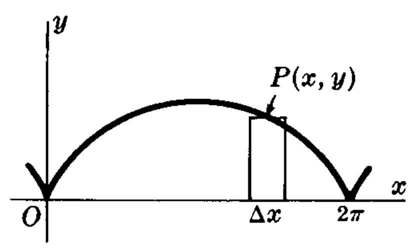

Fig. 39-16

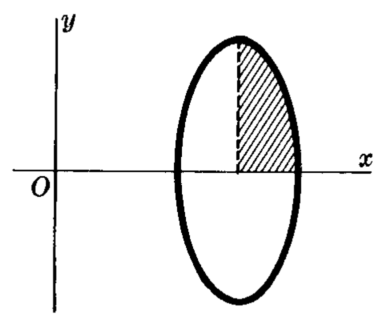

Fig. 39-17

14. Find the area bounded by the curve $x = 3 + \cos\theta$, $y = 4\sin\theta$. (See Fig. 39-17.)

The boundary of the shaded area in the figure (one-quarter of the required area) is described from *right to left* as θ varies from 0 to $\frac{1}{2}\pi$. Hence,

$$A = -4\int_{\theta=0}^{\theta=\pi/2} y\, dx = -4\int_0^{\pi/2} (4\sin\theta)(-\sin\theta)\, d\theta = 16\int_0^{\pi/2} \sin^2\theta\, d\theta = 8\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= 8\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 4\pi \text{ square units}$$

Supplementary Problems

15. Find the area bounded by the given curves, or as described.

(*a*) $y = x^2$, $y = 0$, $x = 2$, $x = 5$
(*b*) $y = x^3$, $y = 0$, $x = 1$, $x = 3$
(*c*) $y = 4x - x^2$, $y = 0$, $x = 1$, $x = 3$
(*d*) $x = 1 + y^2$, $x = 10$
(*e*) $x = 3y^2 - 9$, $x = 0$, $y = 0$, $y = 1$
(*f*) $x = y^2 + 4y$, $x = 0$

(*g*) $y = 9 - x^2$, $y = x + 3$
(*h*) $y = 2 - x^2$, $y = -x$
(*i*) $y = x^2 - 4$, $y = 8 - 2x^2$
(*j*) $y = x^4 - 4x^2$, $y = 4x^2$
(*k*) A loop of $y^2 = x^2(a^2 - x^2)$
(*l*) The loop of $9ay^2 = x(3a - x)^2$
(*m*) $y = e^x$, $y = e^{-x}$, $x = 0$, $x = 2$
(*n*) $y = e^{x/a} + e^{-x/a}$, $y = 0$, $x = \pm a$
(*o*) $xy = 12$, $y = 0$, $x = 1$, $x = e^2$
(*p*) $y = 1/(1 + x^2)$, $y = 0$, $x = \pm 1$
(*q*) $y = \tan x$, $x = 0$, $x = \frac{1}{4}\pi$
(*r*) A circular sector of radius r and angle α
(*s*) Within the ellipse $x = a \cos t$, $y = b \sin t$
(*t*) $x = 2\cos\theta - \cos 2\theta - 1$, $y = 2\sin\theta - \sin 2\theta$
(*u*) $x = a\cos^3 t$, $y = a \sin^3 t$
(*v*) First arch of $y = e^{-ax} \sin ax$
(*w*) $y = xe^{-x^2}$, $y = 0$, and the maximum ordinate
(*x*) The two branches of $(2x - y)^2 = x^3$ and $x = 4$
(*y*) Within $y = 25 - x^2$, $256x = 3y^2$, $16y = 9x^2$

Ans. (all in square units): (*a*) 39; (*b*) 20; (*c*) $\frac{22}{3}$; (*d*) 36; (*e*) 8; (*f*) $\frac{32}{3}$; (*g*) $\frac{125}{6}$; (*h*) $\frac{9}{2}$; (*i*) 32; (*j*) $512\sqrt{2}/15$; (*k*) $2a^3/3$; (*l*) $8\sqrt{3}a^2/5$; (*m*) $(e^2 + 1/e^2 - 2)$; (*n*) $2a(e - 1/e)$; (*o*) 24; (*p*) $\frac{1}{2}\pi$; (*q*) $\frac{1}{2}\ln 2$; (*r*) $\frac{1}{2}r^2$; (*s*) πab; (*t*) 6π; (*u*) $3\pi a^2/8$; (*v*) $(1 + 1/e^{\pi})/2a$; (*w*) $\frac{1}{2}(1 - 1/\sqrt{e})$; (*x*) $\frac{128}{5}$; (*y*) $\frac{98}{3}$

By the *average ordinate* of the curve $y = f(x)$ over the interval $a \le x \le b$ is meant the quantity

$$\frac{\text{area}}{\text{base}} = \frac{\int_a^b f(x)\,dx}{b - a}.$$

16. Find the average ordinate (*a*) of a semicircle of radius; (*b*) of the parabola $y = 4 - x^2$ from $x = -2$ to $x = 2$. *Ans.* (*a*) $\pi r/4$; (*b*) $8/3$

17. (*a*) Find the average ordinate of an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ with respect to x.
(*b*) Repeat part (*a*), with respect to θ.

Ans. (*a*) $\frac{1}{2\pi a}\int_0^{2\pi} a^2(1 - \cos\theta)^2\,d\theta = \frac{3a}{2}$; (*b*) $\frac{1}{2\pi}\int_0^{2\pi} a(1 - \cos\theta)\,d\theta = a$

18. For a freely falling body, $s = \frac{1}{2}gt^2$ and $v = gt = \sqrt{2gs}$.
(*a*) Show that the average value of v with respect to t for the interval $0 \le t \le t_1$ is one-half the final velocity.
(*b*) Show that the average value of v with respect to s for the interval $0 \le s \le s_1$ is two-thirds the final velocity.

19. Prove that (*39.1*) holds when the curves may lie partially or completely below the x axis, as in Fig. 39-3.

Chapter 40

Exponential and Logarithmic Functions; Exponential Growth and Decay

THE NATURAL LOGARITHM. A more rigorous definition of the natural logarithm than that given in Chapter 19 is based on integration.

Definition 40.1: $\ln x = \int_1^x \frac{1}{t}\, dt$, for $x > 0$.

Thus, for $x > 1$, $\ln x$ is the area under the curve $y = 1/t$ between 1 and x, that is, the shaded area in Fig. 40-1.

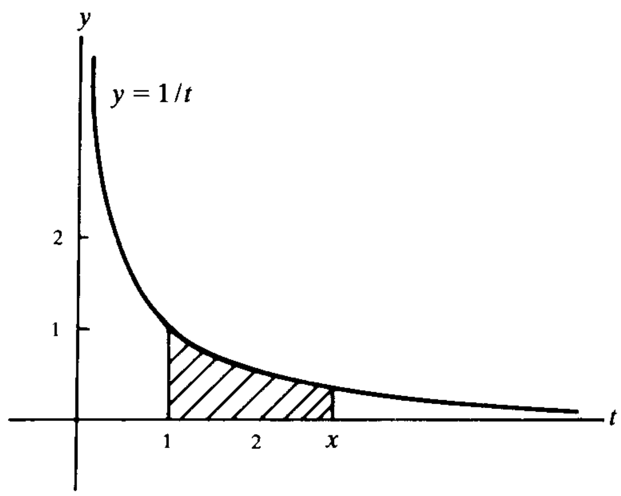

Fig. 40-1

PROPERTIES OF NATURAL LOGARITHMS

40.1. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$

40.2. $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$ for $x \neq 0$

40.3. $\int \frac{1}{x}\, dx = \ln |x| + C$ for $x \neq 0$

40.4. $\ln 1 = 0$

40.5. $\ln x$ is an increasing function. (Hence, if $\ln u = \ln v$, then $u = v$.)

40.6. $\ln 2 > \frac{1}{2}$

40.7. $\ln uv = \ln u + \ln v$

40.8. $\ln \frac{u}{v} = \ln u - \ln v$

40.9. $\ln \frac{1}{v} = -\ln v$

40.10. $\ln u^r = r \ln u$ for all rational numbers r

40.11. $\lim_{x \to +\infty} (\ln x) = +\infty$

40.12. $\lim_{x \to 0^+} (\ln x) = -\infty$

40.13. For each real number y, there is a unique positive number x such that $\ln x = y$.

(See Problems 1 to 6.)

DEFINITIONS

Definition 40.2: e is the unique positive number such that $\ln e = 1$.

Definition 40.3: Let a be greater than zero, and let x be any real number. Then a^x is the unique positive number such that $\ln a^x = x \ln a$.

Definition 40.4: Let a be greater than zero. Then $\log_a x = \dfrac{\ln x}{\ln a}$ for $x > 0$.

PROPERTIES OF a^x AND e^x

40.14. $a^0 = 1$	40.15. $a^1 = a$
40.16. $a^{u+v} = a^u a^v$	40.17. $a^{u-v} = \dfrac{a^u}{a^v}$
40.18. $(a^u)^v = a^{uv}$	40.19. $(ab)^u = a^u b^u$
40.20. $\ln e^x = x$	40.21. $e^{\ln x} = x$

(See Problems 7 to 9.)

DERIVATIVES AND INTEGRALS involving a^x and e^x:

$$\frac{d}{dx}(a^x) = (\ln a)a^x \tag{40.1}$$

$$\frac{d}{dx}(e^x) = e^x \tag{40.2}$$

$$\int e^x\,dx = e^x + C \tag{40.3}$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + C \tag{40.4}$$

(See Problem 10.)

EXPONENTIAL GROWTH AND DECAY. Assume that a quantity y varies with time and $\dfrac{dy}{dt} = ky$ for some nonzero constant k. Then:

$$y = y_0 e^{kt} \qquad \text{where} \qquad y_0 = y(0) \tag{40.5}$$

If $k > 0$, we say that y *grows exponentially* with *growth constant* k. If $k < 0$, we say that y *decays exponentially* with *decay constant* k.

If a substance decays exponentially with decay constant k, then its *halflife* T is the time required for half a given quantity of the substance to disappear, that is, such that $y(T) = \frac{1}{2}y_0$. Then

$$kT = -\ln 2 \tag{40.6}$$

(See Problems 11 to 14.)

Solved Problems

1. Prove Properties 40.1 and 40.2.

Property 40.1 follows from the fact that $\dfrac{d}{dx}(\ln x) = \dfrac{d}{dx}\left(\displaystyle\int_1^x \frac{1}{t}\,dx\right)$ by definition, and that the right-hand side is equal to $1/x$ by Property 38.5.

When $x<0$, $\frac{d}{dx}(\ln|x|)=\frac{d}{dx}(\ln(-x))=\frac{1}{-x}\frac{d}{dx}(-x)=\frac{1}{-x}(-1)=\frac{1}{x}$.

2. Prove Property 40.5.

$\frac{d}{dx}(\ln x)=\frac{1}{x}>0$. Hence, $\ln x$ is an increasing function.

3. Prove $\ln 2>\frac{1}{2}$.

For $1<t<2$, we have $\frac{1}{t}>\frac{1}{2}$. Then $\ln 2=\int_1^2 \frac{1}{t}\,dt>\int_1^2 \frac{1}{2}\,dt=\frac{1}{2}$.

4. Prove Property 40.7: $\ln uv=\ln u+\ln v$.

We have $\frac{d}{dx}(\ln ax)=\frac{1}{ax}a=\frac{1}{x}=\frac{d}{dx}(\ln x)$. Hence, $\ln ax=\ln x+C$.
When $x=1$, $\ln a=\ln 1+C=0+C=C$. Hence, $\ln ax=\ln x+\ln a$. Now let $u=x$ and $v=a$, and Property 40.7 follows.

5. Prove Property 40.10: $\ln a^r=r\ln a$ for rational r.

We have $\frac{d}{dx}(\ln x^r)=\frac{1}{x^r}(rx^{r-1})=\frac{r}{x}=\frac{d}{dx}(r\ln x)$. Hence, $\ln x^r=r\ln x+C$.
When $x=1$, this becomes $\ln 1^r=\ln 1=0=r\ln 1+C=C$. Thus, $C=0$ and $\ln x^r=r\ln x$.

6. Prove Property 40.11: $\lim_{x\to+\infty}\ln x=+\infty$.

Given any positive integer N, choose $x=2^{2N}$. Then $\ln x=\ln 2^{2N}=2N\ln 2>N$ by Property 40.6. Since $\ln x$ is increasing, $\ln x>N$ for all $x\geq 2^{2N}$.

7. Prove Properties 40.14 and 40.15.

By definition, $\ln a^0=0\ln a=0=\ln 1$. Hence Property 40.14: $a^0=1$.
By definition, $\ln a^1=1\ln a=\ln a$. Hence Property 40.15: $a^1=a$.

8. Prove Property 40.16.

$$\ln a^{u+v}=(u+v)\ln a=u\ln a+v\ln a=\ln a^u+\ln a^v=\ln(a^u a^v)$$

Hence, $a^{u+v}=a^u a^v$.

9. Prove Properties 40.20 and 40.21.

For Property 40.20: $\ln e^x=x\ln e=x\cdot 1=x$.
For Property 40.21: $\ln e^{\ln x}=\ln x\ln e=\ln x$. Hence, $e^{\ln x}=x$.

10. Assuming that $y=a^x$ is differentiable, show that $\frac{d}{dx}(a^x)=a^x\ln a$.

Let $y=a^x$. Then $\ln y=\ln a^x=x\ln a$. Differentiate to obtain

$$\frac{1}{y}\frac{dy}{dx}=\ln a \qquad \text{from which} \qquad \frac{dy}{dx}=y\ln a=a^x\ln a$$

11. Show that, if $\frac{dy}{dt}=ky$, then $y=y_0e^{kt}$, where $y_0=y(0)$.

$$\frac{d}{dt}\left(\frac{y}{e^{kt}}\right)=\frac{e^{kt}(dy/dt)-kye^{kt}}{e^{2kt}}=\frac{e^{kt}(ky)-kye^{kt}}{e^{2kt}}=0$$

Hence $\frac{y}{e^{kt}} = C$, so $y = Ce^{kt}$. Now $y_0 = y(0) = Ce^0 = C$, so that $y = y_0e^{kt}$.

12. Prove the relation $kT = -\ln 2$ between the decay constant and the halflife T.

By the definition of halflife, $y_0/2 = y_0e^{kT}$, or $\frac{1}{2} = e^{kT}$. Then $\ln \frac{1}{2} = \ln e^{kT} = kT$. But $\ln \frac{1}{2} = -\ln 2$, proving the relation.

13. If 20% of a radioactive substance disappears in one year, find its halflife. Assume exponential decay.

By (*40.5*), $0.8y_0 = y_0e^k$. So $0.8 = e^k$, from which $k = \ln 0.8 = \ln \frac{4}{5} = \ln 4 - \ln 5$. Then (*40.6*) yields $T = -\frac{\ln 2}{k} = \frac{\ln 2}{\ln 5 - \ln 4}$.

14. If the number of bacteria in a culture grows exponentially with a growth constant of 0.02, with time measured in hours, how many bacteria will be present in one hour if there are initially 1000?

From (*40.5*), $y = 1000e^{0.02} \approx 1000(1.0202) = 1020.2 \approx 1020$.

Supplementary Problems

15. Prove Properties 40.8, 40.9, 40.12, and 40.13.

16. Prove Properties 40.17 to 40.19.

17. Prove the following properties of logarithms to the base a:
(*a*) $\log_a 1 = 0$ (*b*) $\log_a uv = \log_a u + \log_a v$ (*c*) $\log_a \frac{u}{v} = \log_a u - \log_a v$
(*d*) $\log_a u^r = r \log_a u$ (*e*) $\log_a \frac{1}{v} = -\log_a v$ (*f*) $a^{\log_a x} = x$

18. Assume that, in a chemical reaction, a certain substance decomposes at a rate proportional to the amount present. In 5 hours, an initial quantity of 10,000 grams is reduced to 1000 grams. How much will be left of an initial quantity of 20,000 grams after 15 hours? *Ans.* 20 grams

19. A container with a maximum capacity of 25,000 fruit flies initially contains 1000 fruit flies. If the population grows exponentially with a growth constant of $\frac{\ln 5}{10}$ fruit flies per day, in how many days will the container be full? *Ans.* 20 days

20. The halflife of radium is 1690 years. How much will be left of 32 grams of radium after 6760 years? *Ans.* 2 grams

21. A saltwater solution initially contains 5 lb of salt in 10 gal of fluid. If water flows in at the rate of $\frac{1}{2}$ gal/min and the mixture flows out at the same rate, how much salt is present after 20 min?

Ans. $dS/dt = -\frac{1}{2}(S/10)$; at $t = 20$, $S = 5/e \approx 1.8395$ lb

22. Assume that a population grows exponentially and increases at the rate of $K\%$ per year. (*a*) Find its growth constant k. (*b*) Approximate k when $K = 2$.

Ans. (*a*) $k = \ln(1 + K/100)$; (*b*) $k \approx 0.0198$

Chapter 41

Volumes of Solids of Revolution

A SOLID OF REVOLUTION is generated by revolving a plane area about a line, called the *axis of rotation*, in the plane. The *volume* of a solid of revolution may be found with one of the following procedures.

DISC METHOD. This method is useful when the axis of rotation is part of the boundary of the plane area.

1. Make a sketch showing the area involved, a representative strip perpendicular to the axis of rotation, and the approximating rectangle, as in Chapter 39.
2. Write the volume of the disc (or cylinder) generated when the approximating rectangle is revolved about the axis of rotation, and sum for the n rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

When the axis of rotation is the x axis and the top of the plane area is given by the curve $y = f(x)$ between $x = a$ and $x = b$ (Fig. 41-1), then the volume V of the solid of revolution is given by

$$V = \int_a^b \pi y^2\, dx = \pi \int_a^b [f(x)]^2\, dx \qquad (41.1)$$

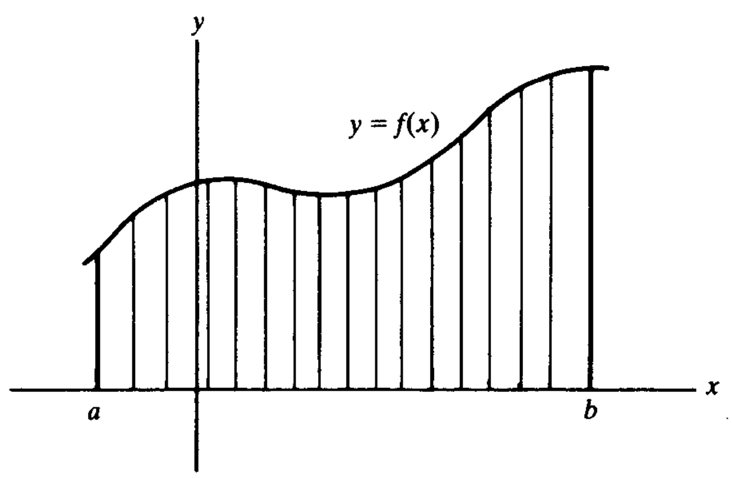

Fig. 41-1

Similarly, when the axis of rotation is the y axis and one side of the plane area is given by the curve $x = g(y)$ between $y = c$ and $y = d$ (Fig. 41-2), then the volume V of the solid of revolution is given by

$$V = \int_c^d \pi x^2\, dy = \pi \int_c^d [g(y)]^2\, dy \qquad (41.2)$$

(See Problems 1 and 2.)

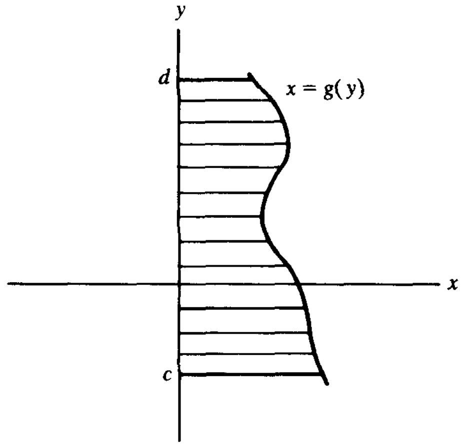

Fig. 41-2

WASHER METHOD. This method is useful when the axis of rotation is not a part of the boundary of the plane area.

1. Same as step 1 of the disc method.
2. Extend the sides of the approximating rectangle $ABCD$ to meet the axis of rotation in E and F, as in Fig. 41-9. When the approximating rectangle is revolved about the axis of rotation, a washer is formed whose volume is the difference between the volumes generated by revolving the rectangles $EABF$ and $ECDF$ about the axis. Write the difference of the two volumes, and proceed as in step 2 of the disc method.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

If the axis of rotation is the x axis, the upper boundary of the plane area is given by $y = f(x)$, the lower boundary by $y = g(x)$, and the region runs from $x = a$ to $x = b$ (Fig. 41-3), then the volume V of the solid of revolution is given by

$$V = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}\, dx \tag{41.3}$$

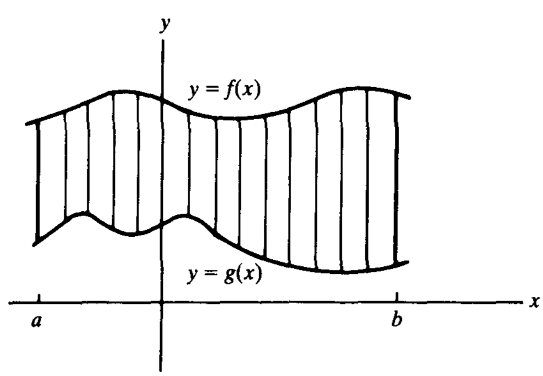

Fig. 41-3

Similarly, if the axis of rotation is the y axis and the plane area is bounded to the right by $x = f(y)$, to the left by $x = g(y)$, above by $y = d$, and below by $y = c$ (Fig. 41-4), then the volume V is given by

$$V = \pi \int_c^d \{[f(y)]^2 - [g(y)]^2\}\, dy \tag{41.4}$$

(See Problems 3 and 4.)

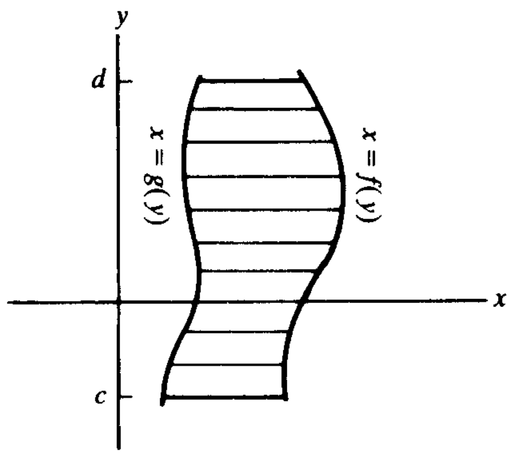

Fig. 41-4

SHELL METHOD

1. Make a sketch showing the area involved, a representative strip parallel to the axis of rotation, and the approximating rectangle.
2. Write the volume (= mean circumference × height × thickness) of the cylindrical shell generated when the approximating rectangle is revolved about the axis of rotation, and sum for the n rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

If the axis of rotation is the y axis and the plane area, in the first quadrant, is bounded below by the x axis, above by $y = f(x)$, to the left by $x = a$, and to the right by $x = b$ (Fig. 41-5), then the volume V is given by

$$V = 2\pi \int_a^b xy\, dx = 2\pi \int_a^b xf(x)\, dx \tag{41.5}$$

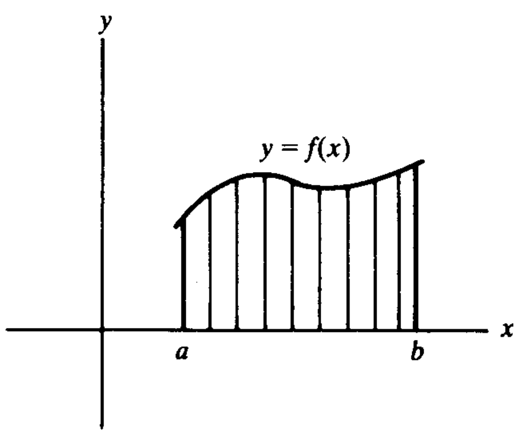

Fig. 41-5

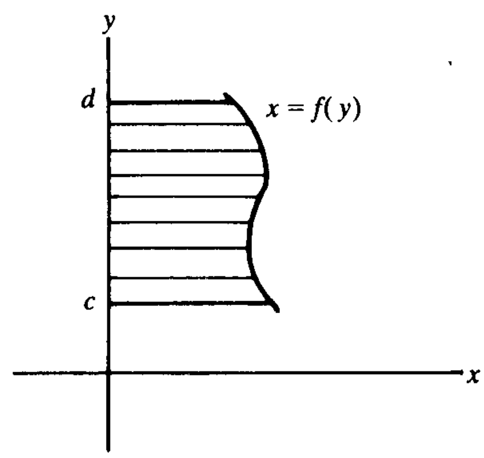

Fig. 41-6

Similarly, if the axis of rotation is the x axis and the plane area, in the first quadrant, is bounded to the left by the y axis, to the right by $x = f(y)$, below by $y = c$, and above by $y = d$ (Fig. 41-6), then the volume V is given by

$$V = 2\pi \int_c^d xy\,dy = 2\pi \int_c^d yf(y)\,dy \tag{41.6}$$

(See Problems 5 to 8.)

Solved Problems

1. Find the volume generated by revolving the first-quadrant area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the x axis.

We divide the plane area vertically, as can be seen in Fig. 41-7. When the approximating rectangle is revolved about the x axis, a disc whose radius is y, whose height is Δx, and whose volume is $\pi y^2\,\Delta x$ is generated. The sum of the volumes of n discs, corresponding to the n approximating rectangles, is $\Sigma\,\pi y^2\,\Delta x$, and the required volume is

$$V = \int_a^b dV = \int_0^2 \pi y^2\,dx = \pi \int_0^2 8x\,dx = 4\pi x^2\Big|_0^2 = 16\pi \text{ cubic units}$$

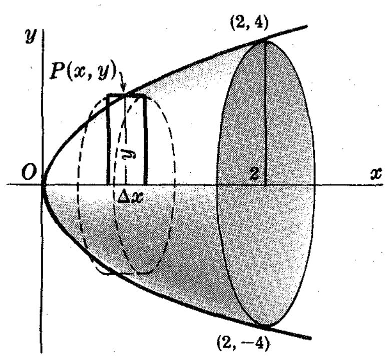

Fig. 41-7

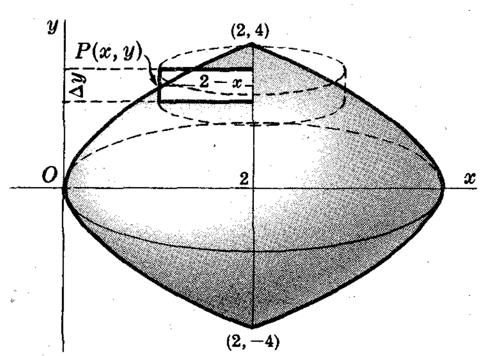

Fig. 41-8

2. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the latus rectum.

We divide the area horizontally, as can be seen in Fig. 41-8. When the approximating rectangle is revolved about the latus rectum, it generates a disc whose radius is $2 - x$, whose height is Δy, and whose volume is $\pi(2 - x)^2\,\Delta y$. The required volume is then

$$V = \int_{-4}^4 \pi(2 - x)^2\,dy = 2\pi \int_0^4 (2 - x)^2\,dy = 2\pi \int_0^4 \left(2 - \frac{y^2}{8}\right)^2 dy = \frac{256}{15}\pi \text{ cubic units}$$

3. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum $(x = 2)$ about the y axis.

We divide the area horizontally, as shown in Fig. 41-9. When the approximating rectangle is revolved about the y axis, it generates a washer whose volume is the difference between the volumes

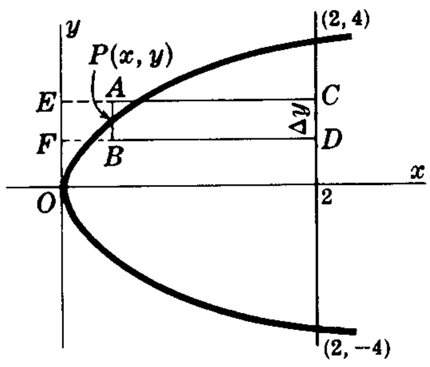

Fig. 41-9

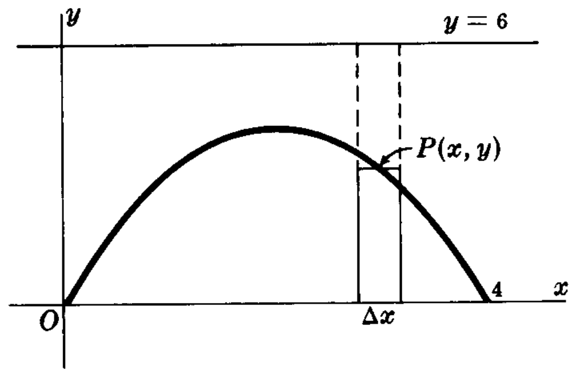

Fig. 41-10

generated by revolving the rectangle $ECDF$ (of dimensions 2 by Δy) and the rectangle $EABF$ (of dimensions x by Δy) about the y axis, that is, $\pi(2)^2\,\Delta y - \pi(x)^2\,\Delta y$. The required volume is then

$$V = \int_{-4}^{4} 4\pi\,dy - \int_{-4}^{4} \pi x^2\,dy = 2\pi \int_0^4 (4 - x^2)\,dy = 2\pi \int_0^4 \left(4 - \frac{y^2}{64}\right) dy = \frac{128}{5}\pi \text{ cubic units}$$

4. Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the x axis about the line $y = 6$.

We divide the area vertically (Fig. 41-10). The solid generated by revolving the approximating rectangle about the line $y = 6$ is a washer whose volume is $\pi(6)^2\,\Delta x - \pi(6 - y)^2\,\Delta x$. The required volume is then

$$V = \pi \int_0^4 [(6)^2 - (6 - y)^2]\,dx = \pi \int_0^4 (12y - y^2)\,dx$$

$$= \pi \int_0^4 (48x - 28x^2 + 8x^3 - x^4)\,dx = \frac{1408\pi}{15} \text{ cubic units}$$

5. Justify *(41.5)*.

Refer to Fig. 41-11. Suppose the volume in question is generated by revolving about the y axis the first-quadrant area under the curve $y = f(x)$ from $x = a$ to $x = b$. Let this area be divided into n strips, and each strip be approximated by a rectangle. When the representative rectangle is revolved about the y axis, a cylindrical shell of height y_k, inner radius ξ_{k-1}, outer radius ξ_k, and volume

$$\Delta_k V = \pi(\xi_k^2 - \xi_{k-1}^2)y_k \tag{1}$$

is generated. By the law of the mean for derivatives,

$$\xi_k^2 - \xi_{k-1}^2 = \left[\frac{d}{dx}(x^2)\right]_{x=X_k} (\xi_k - \xi_{k-1}) = 2X_k\,\Delta_k x \tag{2}$$

where $\xi_{k-1} < X_k < \xi_k$. Then *(1)* becomes

$$\Delta_k V = 2\pi X_k y_k\,\Delta_k x = 2\pi X_k f(x_k)\,\Delta_k x$$

and, by the theorem of Bliss,

$$V = 2\pi \lim_{n\to+\infty} \sum_{k=1}^{n} X_k f(x_k)\,\Delta_k x = 2\pi \int_a^b x f(x)\,dx$$

Note: If the policy of choosing the points x_k as the midpoints of the subintervals, used in the preceding chapter, is followed, the theorem of Bliss is not needed. For, by Problem 17(*b*) of Chapter 26, the X_k defined by *(2)* is then $X_k = \frac{1}{2}(\xi_k + \xi_{k-1}) = x_k$. Thus, the volume generated by revolving the n rectangles about the y axis is $\sum_{k=1}^{n} 2\pi x_k f(x_k)\,\Delta_k x = \sum_{k=1}^{n} g(x_k)\,\Delta_k x$, of the type *(38.1)*.

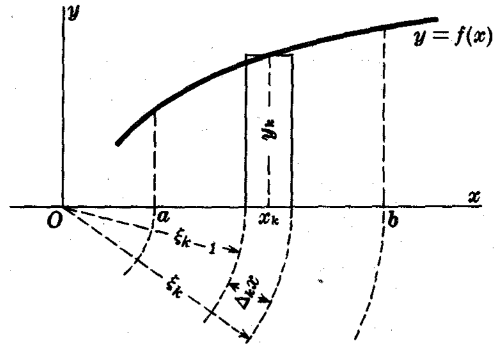

Fig. 41-11

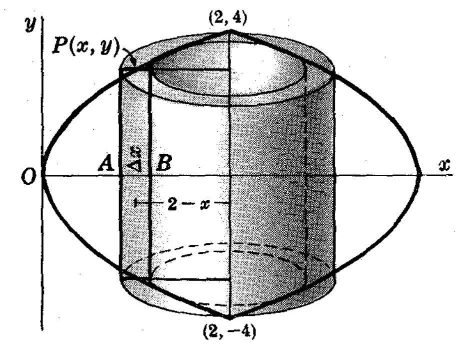

Fig. 41-12

6. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum about the latus rectum. Use the shell method. (See Problem 2.)

We divide the area vertically (Fig. 41-12) and, for convenience, choose the point P so that x is the midpoint of the segment AB. The approximating rectangle has height $2y = 4\sqrt{2x}$ and width Δx, and its mean distance from the latus rectum is $2 - x$. When the rectangle is revolved about the latus rectum, the volume of the cylindrical shell generated is $2\pi(2-x)(4\sqrt{2x}\,\Delta x)$. The required volume is then

$$V = 8\sqrt{2}\pi \int_0^2 (2-x)\sqrt{x}\,dx = 8\sqrt{2}\pi \int_0^2 (2x^{1/2} - x^{3/2})\,dx = \frac{256\pi}{15} \text{ cubic units}$$

7. Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$.

We shall use the shell method (Fig. 41-13). The approximating rectangle is of height $2y$, thickness Δx, and mean distance from the axis of revolution $3 - x$. The required volume is then

$$V = 2\pi \int_{-2}^{2} 2y(3-x)\,dx = 4\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\,dx = 12\pi \int_{-2}^{2} \sqrt{4-x^2}\,dx - 4\pi \int_{-2}^{2} x\sqrt{4-x^2}\,dx$$

$$= \left[12\pi\left(\frac{x}{2}\sqrt{4-x^4} + 2\arcsin\frac{x}{2}\right) + \frac{4\pi}{3}(4-x^2)^{3/2}\right]_{-2}^{2} = 24\pi^2 \text{ cubic units}$$

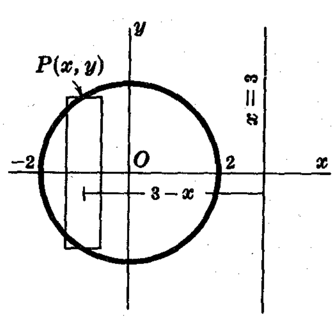

Fig. 41-13

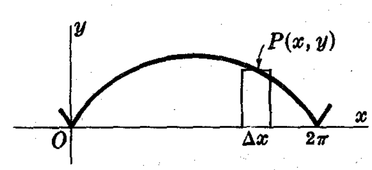

Fig. 41-14

8. Find the volume of the solid generated by revolving about the y axis the area between the first arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ and the x axis. Use the shell method.

From Fig. 41-14,

$$V = 2\pi \int_{\theta=0}^{\theta=2\pi} xy\,dx = 2\pi \int_0^{2\pi} (\theta - \sin\theta)(1-\cos\theta)(1-\cos\theta)\,d\theta$$

$$= 2\pi \int_0^{2\pi} (\theta - 2\theta\cos\theta + \theta\cos^2\theta - \sin\theta + 2\sin\theta\cos\theta - \cos^2\theta\sin\theta)\,d\theta$$

$$= 2\pi\left[\tfrac{3}{4}\theta^2 - 2(\theta\sin\theta + \cos\theta) + \tfrac{1}{2}\left(\tfrac{1}{2}\theta\sin 2\theta + \tfrac{1}{4}\cos 2\theta\right) + \cos\theta + \sin^2\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{2\pi}$$

$$= 6\pi^3 \text{ cubic units}$$

9. Find the volume generated when the plane area bounded by $y = -x^2 - 3x + 6$ and $x + y - 3 = 0$ is revolved (*a*) about $x = 3$, and (*b*) about $y = 0$.

From Fig. 41-15,

(*a*) $V = 2\pi \int_{-3}^{1} (y_C - y_L)(3 - x)\,dx = 2\pi \int_{-3}^{1} (x^3 - x^2 - 9x + 9)\,dx = \dfrac{256\pi}{3}$ cubic units

(*b*) $V = \pi \int_{-3}^{1} y_C^2 - y_L^2\,dx = \pi \int_{-3}^{1} (x^4 + 6x^3 - 4x^2 - 30x + 27)\,dx = \dfrac{1792\pi}{15}$ cubic units

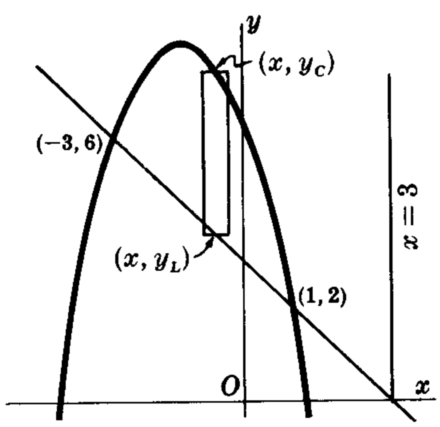

Fig. 41-15

Supplementary Problems

In Problems 10 to 19, find the volume generated by revolving the given plane area about the given line, using the disc method. (Answers are in cubic units.)

10. Within $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about x axis *Ans.* 2500π

11. Within $x^2 - y^2 = 16$, $y = 0$, $x = 8$; about x axis *Ans.* $256\pi/3$

12. Within $y = 4x^2$, $x = 0$, $y = 16$; about y axis *Ans.* 32π

13. Within $y = 4x^2$, $x = 0$, $y = 16$; about $y = 16$ *Ans.* $4096\pi/15$

14. Within $y^2 = x^3$, $y = 0$, $x = 2$; about x axis *Ans.* 4π

15. Within $y = x^3$, $y = 0$, $x = 2$; about $x = 2$ *Ans.* $16\pi/5$

16. Within $y^2 = x^4(1 - x^2)$; about x axis *Ans.* $4\pi/35$

17. Within $4x^2 + 9y^2 = 36$; about x axis *Ans.* 16π

18. Within $4x^2 + 9y^2 = 36$, about y axis *Ans.* 24π

19. Within $x = 9 - y^2$, between $x - y - 7 = 0$, $x = 0$; about y axis *Ans.* $963\pi/5$

In Problems 20 to 26, find the volume generated by revolving the given plane area about the given line, using the washer method. (Answers are in cubic units.)

20. Within $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about y axis *Ans.* 625π

21. Within $x^2 - y^2 = 16$, $y = 0$, $x = 8$; about y axis *Ans.* $128\sqrt{3}\pi$

22. Within $y = 4x^2$, $x = 0$, $y = 16$; about x axis *Ans.* $2048\pi/5$

23. Within $y = x^3$, $x = 0$, $y = 8$; about $x = 2$ *Ans.* $144\pi/5$

24. Within $y = x^2$, $y = 4x - x^2$; about x axis *Ans.* $32\pi/3$

25. Within $y = x^2$, $y = 4x - x^2$; about $y = 6$ *Ans.* $64\pi/3$

26. Within $x = 9 - y^2$, $x - y - 7 = 0$; about $x = 4$ *Ans.* $153\pi/5$

In Problems 27 to 32, find the volume generated by revolving the given plane area about the given line, using the shell method. (Answers are in cubic units.)

27. Within $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about y axis *Ans.* 625π

28. Within $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about $x = 6$ *Ans.* 375π

29. Within $y = x^3$, $y = 0$, $x = 2$; about $y = 8$ *Ans.* $320\pi/7$

30. Within $y = x^2$, $y = 4x - x^2$; about $x = 5$ *Ans.* $64\pi/3$

31. Within $y = x^2 - 5x + 6$, $y = 0$; about y axis *Ans.* $5\pi/6$

32. Within $x = 9 - y^2$, between $x - y - 7 = 0$, $x = 0$; about $y = 3$ *Ans.* $369\pi/2$

In Problems 33 to 39, find the volume generated by revolving the given plane area about the given line, using any appropriate method. (Answers are in cubic units.)

33. Within $y = e^{-x^2}$, $y = 0$, $x = 0$, $x = 1$; about y axis *Ans.* $\pi(1 - 1/e)$

34. Within an arch of $y = \sin 2x$; about x axis *Ans.* $\frac{1}{4}\pi^2$

35. Within first arch of $y = e^x \sin x$; about x axis *Ans.* $\pi(e^{2\pi} - 1)/8$

36. Within first arch of $y = e^x \sin x$; about y axis *Ans.* $\pi[(\pi - 1)e^{\pi} - 1]$

37. Within first arch of $x = \theta - \sin\theta$, $y = 1 - \cos\theta$; about x axis *Ans.* $5\pi^2$

38. Within the cardioid $x = 2\cos\theta - \cos 2\theta - 1$, $y = 2\sin\theta - \sin 2\theta$; about x axis *Ans.* $64\pi/3$

39. Within $y = 2x^2$, $2x - y + 4 = 0$; about $x = 2$ *Ans.* 27π

40. Obtain the volume of the frustum of a cone whose lower base is of radius R, upper base is of radius r, and altitude is h. *Ans.* $\frac{1}{3}\pi h(r^2 + rR + R^2)$ cubic units

Chapter 42

Volumes of Solids with Known Cross Sections

THE VOLUME OF THE SOLID OF REVOLUTION that is generated by revolving about the x axis the plane area bounded by the curve $y = f(x)$, the x axis, and the lines $x = a$ and $x = b$ is given by $\int_a^b \pi y^2\,dx$. The integrand $\pi y^2 = \pi[f(x)]^2$ may be interpreted as the area of the cross section of the solid made by a plane perpendicular to the x axis and at a distance x units from the origin.

Conversely, assume that the area of a cross section ABC of a solid, made by a plane perpendicular to the x axis at a distance x from the origin, can be expressed as a function $A(x)$ of x. Then the volume of the solid is given by

$$V = \int_\alpha^\beta A(x)\,dx$$

(See Fig. 42-1.) The x coordinates of the points of the solid lie in the interval $\alpha \le x \le \beta$.

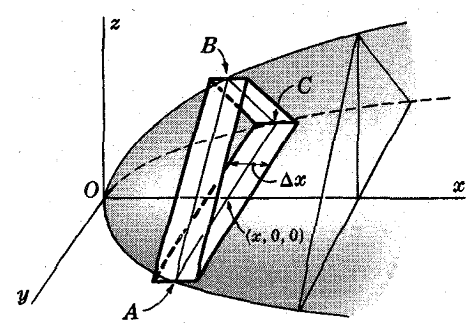

Fig. 42-1

Solved Problems

1. A solid has a circular base of radius 4 units. Find the volume of the solid if every plane section perpendicular to a particular fixed diameter is an equilateral triangle.

Take the circle as in Fig. 42-2, with the fixed diameter on the x axis. The equation of the circle is $x^2 + y^2 = 16$. The cross section ABC of the solid is an equilateral triangle of side $2y$ and area $A(x) = \sqrt{3}y^2 = \sqrt{3}(16 - x^2)$. Then

$$V = \int_\alpha^\beta A(x)\,dx = \sqrt{3}\int_{-4}^{4} (16 - x^2)\,dx = \sqrt{3}\left[16x - \frac{x^3}{3}\right]_{-4}^{4} = \frac{256}{3}\sqrt{3} \text{ cubic units}$$

2. A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Find its volume if every section perpendicular to the major axis is an isosceles triangle with altitude 6.

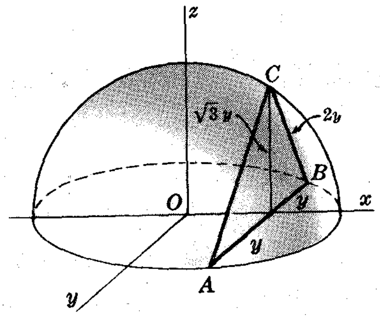

Fig. 42-2

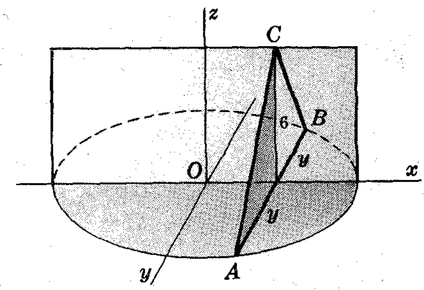

Fig. 42-3

Take the ellipse as in Fig. 42-3, with equation $\frac{x^2}{25}+\frac{y^2}{16}=1$. The section ABC is an isosceles triangle of base $2y$, altitude 6, and area $A(x)=6y=6(\frac{4}{5}\sqrt{25-x^2})$. Hence,

$$V=\frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx=60\pi \text{ cubic units}$$

3. Find the volume of the solid cut from the paraboloid $\frac{x^2}{16}+\frac{y^2}{25}=z$ by the plane $z=10$.

Refer to Fig. 42-4. The section of the solid cut by a plane parallel to the plane xOy and at a distance z from the origin is an ellipse of area $\pi xy=\pi(4\sqrt{z})(5\sqrt{z})=20\pi z$. Hence

$$V=20\pi\int_{0}^{10} z\,dz=1000\pi \text{ cubic units}$$

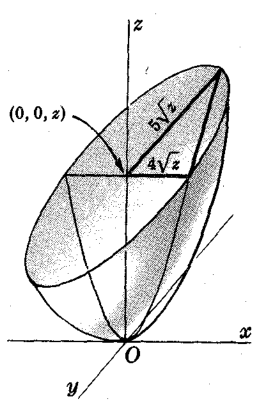

Fig. 42-4

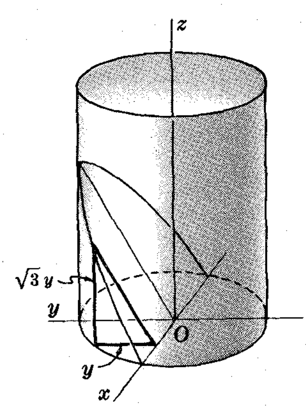

Fig. 42-5

4. Two cuts are made on a circular log of radius 8 inches, the first perpendicular to the axis of the log and the second inclined at the angle of 60° with the first. If the two cuts meet on a line through the center, find the volume of the wood cut out.

Refer to Fig. 42-5. Take the origin at the center of the log, the x axis along the intersection of the two cuts, and the positive side of the y axis in the face of the first cut. A section of the cut made by a plane perpendicular to the x axis is a right triangle having one angle of 60° and the adjacent leg of length

y. The other leg is of length $\sqrt{3}y$, and the area of the section is $\frac{1}{2}\sqrt{3}y^2 = \frac{1}{2}\sqrt{3}(64 - x^2)$. Then

$$V = \frac{1}{2}\sqrt{3}\int_{-8}^{8}(64 - x^2)\,dx = \frac{1024}{3}\sqrt{3}\text{ in}^3$$

5. The axes of two circular cylinders of equal radii r intersect at right angles. Find their common volume.

Refer to Fig. 42-6. Let the cylinders have equations $x^2 + z^2 = r^2$ and $y^2 + z^2 = r^2$. A section of the solid whose volume is required, as cut by a plane perpendicular to the z axis, is a square of side $2x = 2y = 2\sqrt{r^2 - z^2}$ and area $4(r^2 - z^2)$. Hence

$$V = 4\int_{-r}^{r}(r^2 - z^2)\,dz = \frac{16r^3}{3}\text{ cubic units}$$

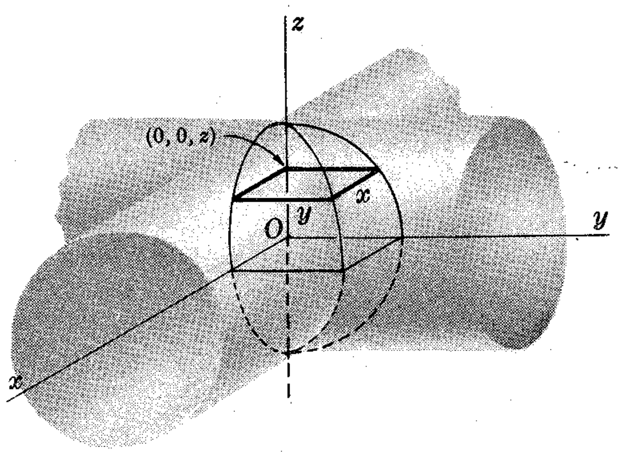

Fig. 42-6

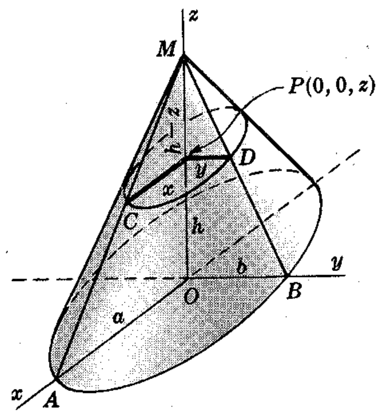

Fig. 42-7

6. Find the volume of the right cone of height h whose base is an ellipse of major axis $2a$ and minor axis $2b$.

A section of the cone cut by a plane parallel to the base is an ellipse of major axis $2x$ and minor axis $2y$ (Fig. 42-7). From similar triangles,

$$\frac{PC}{OA} = \frac{PM}{OM} \quad\text{or}\quad \frac{x}{a} = \frac{h-z}{h} \qquad\text{and}\qquad \frac{PD}{OB} = \frac{PM}{OM} \quad\text{or}\quad \frac{y}{b} = \frac{h-z}{h}$$

The area of the section is thus $\pi xy = \dfrac{\pi ab(h-z)^2}{h^2}$. Hence

$$V = \frac{\pi ab}{h^2}\int_0^h (h-z)^2\,dz = \frac{1}{3}\pi abh\text{ cubic units}$$

Supplementary Problems

7. A solid has a circular base of radius 4 units. Find the volume of the solid if every plane perpendicular to a fixed diameter (the x axis of Fig. 42-2) is (*a*) a semicircle; (*b*) a square; (*c*) an isosceles right triangle with the hypotenuse in the plane of the base.

Ans. (*a*) $128\pi/3$; (*b*) $1024/3$; (*c*) $256/3$ cubic units

8. A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Find its volume if every section perpendicular to the major axis is an isosceles right triangle with one leg in the plane of the base. *Ans.* 640/3 cubic units

9. The base of a solid is the segment of the parabola $y^2 = 12x$ cut off by the latus rectum. A section of the solid perpendicular to the axis of the parabola is a square. Find its volume. *Ans.* 216 cubic units

10. The base of a solid is the first-quadrant area bounded by the line $4x + 5y = 20$ and the coordinate axes. Find its volume if every plane section perpendicular to the x axis is a semicircle.

Ans. $10\pi/3$ cubic units

11. The base of a solid is the circle $x^2 + y^2 = 16x$, and every plane section perpendicular to the x axis is a rectangle whose height is twice the distance of the plane of the section from the origin. Find its volume. *Ans.* 1024π cubic units

12. A horn-shaped solid is generated by moving a circle, having the ends of a diameter on the first-quadrant arcs of the parabolas $y^2 + 8x = 64$ and $y^2 + 16x = 64$, parallel to the xz plane. Find the volume generated. *Ans.* $256\pi/15$ cubic units

13. The vertex of a cone is at $(a, 0, 0)$, and its base is the circle $y^2 + z^2 - 2by = 0$, $x = 0$. Find its volume. *Ans.* $\frac{1}{3}\pi ab^2$ cubic units

14. Find the volume of the solid bounded by the paraboloid $y^2 + 4z^2 = x$ and the plane $x = 4$.

Ans. 4π cubic units

15. A barrel has the shape of an ellipsoid of revolution with equal pieces cut from the ends. Find its volume if its height is 6 ft, its midsection has radius 3 ft, and its ends have radius 2 ft. *Ans.* 44π ft^3

16. The section of a certain solid cut by any plane perpendicular to the x axis is a circle with the ends of a diameter lying on the parabolas $y^2 = 9x$ and $x^2 = 9y$. Find its volume. *Ans.* $6561\pi/280$ cubic units

17. The section of a certain solid cut by any plane perpendicular to the x axis is a square with the ends of a diagonal lying on the parabolas $y^2 = 4x$ and $x^2 = 4y$. Find its volume. *Ans.* 144/35 cubic units

18. A hole of radius 1 inch is bored through a sphere of radius 3 inches, the axis of the hole being a diameter of the sphere. Find the volume of the sphere which remains. *Ans.* $64\pi\sqrt{2}/3$ in^3

Chapter 43

Centroids of Plane Areas and Solids of Revolution

THE MASS OF A PHYSICAL BODY is a measure of the quantity of matter in it, whereas the volume of the body is a measure of the space it occupies. If the mass per unit volume is the same throughout, the body is said to be *homogeneous* or to have *constant density*.

It is highly desirable in physics and mechanics to consider a given mass as concentrated at a point, called its center of mass (also, its center of gravity). For a homogeneous body, this point coincides with its geometric center or *centroid*. For example, the center of mass of a homogeneous rubber ball coincides with the centroid (center) of the ball considered as a geometric solid (a sphere).

The centroid of a rectangular sheet of paper lies midway between the two surfaces but it may well be considered as located on one of the surfaces at the intersection of the diagonals. Then the center of mass of a thin sheet coincides with the centroid of the sheet considered as a plane area.

The discussion in this and the next chapter will be limited to plane areas and solids of revolution. Other solids, the arc of a curve (a piece of fine homogeneous wire), and nonhomogeneous masses will be treated in later chapters.

THE (FIRST) MOMENT M_L OF A PLANE AREA with respect to a line L is the product of the area and the directed distance of its centroid from the line. The moment of a composite area with respect to a line is the sum of the moments of the individual areas with respect to the line.

The moment of a plane area with respect to a coordinate axis may be found as follows:

1. Sketch the area, showing a representative strip and the approximating rectangle.
2. Form the product of the area of the rectangle and the distance of its centroid from the axis, and sum for all the rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

(See Problem 2.)

For a plane area A having centroid $(\bar{x}, \bar{y})$ and moments M_z and M_y with respect to the x and y axes,

$$A\bar{x} = M_y \qquad \text{and} \qquad A\bar{y} = M_x$$

(See Problems 1 to 8.)

THE (FIRST) MOMENT OF A SOLID of volume V, generated by revolving a plane area about a coordinate axis, with respect to the plane through the origin and perpendicular to the axis may be found as follows:

1. Sketch the area, showing a representative strip and the approximating rectangle.
2. Form the product of the volume, disc, or shell generated by revolving the rectangle about the axis and the distance of the centroid of the rectangle from the plane, and sum for all the rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

When the area is revolved about the x axis, the centroid $(\bar{x}, \bar{y})$ is on that axis. If M_{yz} is the

moment of the solid with respect to the plane through the origin and perpendicular to the x axis, then

$$V\bar{x} = M_{yz} \qquad \text{and} \qquad \bar{y} = 0$$

Similarly, when the area is revolved about the y axis, the centroid $(\bar{x}, \bar{y})$ is on that axis. If M_{xz} is the moment of the solid with respect to the plane through the origin and perpendicular to the y axis, then

$$V\bar{y} = M_{xz} \qquad \text{and} \qquad \bar{x} = 0$$

(See Problems 9 to 12.)

FIRST THEOREM OF PAPPUS. If a plane area is revolved about an axis in its plane and not crossing the area, then the volume of the solid generated is equal to the product of the area and the length of the path described by the centroid of the area. (See Problems 13 to 15.)

Solved Problems

1. For the plane area shown in Fig. 43-1, find (*a*) the moments with respect to the coordinate axes and (*b*) the coordinates of the centroid $(\bar{x}, \bar{y})$.

(*a*) The upper rectangle has area $5 \times 2 = 10$ units and centroid $A(2.5, 9)$. Similarly, the areas and centroids of the other rectangles are: 12 units, $B(1, 5)$; 2 units, $C(2.5, 5)$; 10 units, $D(2.5, 1)$.

The moments of these rectangles with respect to the x axis are, respectively, 10(9), 12(5), 2(5), and 10(1). Hence the moment of the figure with respect to the x axis is $M_x = 10(9) + 12(5) + 2(5) + 10(1) = 170$.

Similarly, the moment of the figure with respect to the y axis is $M_y = 10(2.5) + 12(1) + 2(2.5) + 10(2.5) = 67$.

(*b*) The area of the figure is $A = 10 + 12 + 2 + 10 = 34$. Since $A\bar{x} = M_y$, $34\bar{x} = 67$ and $\bar{x} = \frac{67}{34}$. Also, since $A\bar{y} = M_x$, $34\bar{y} = 170$ and $\bar{y} = 5$. Hence the point $(\frac{67}{34}, 5)$ is the centroid.

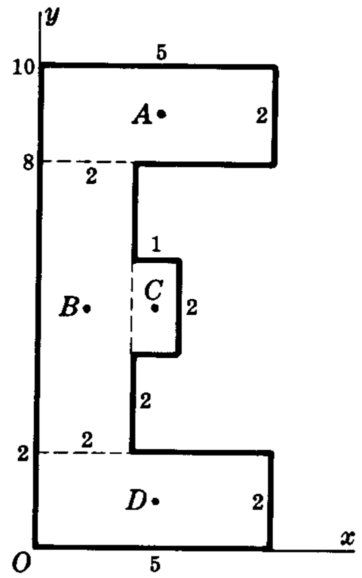

Fig. 43-1

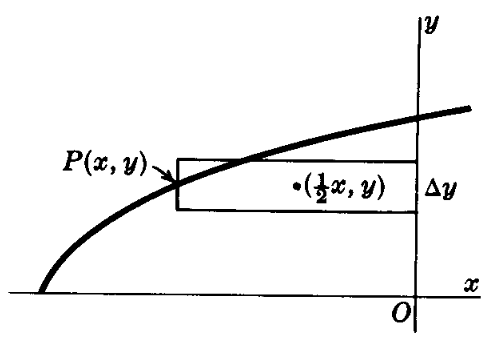

Fig. 43-2

2. Find the moments with respect to the coordinate axes of the plane area in the second quadrant bounded by the curve $x = y^2 - 9$.

We use the approximating rectangle shown in Fig. 43-2. Its area is $-x\,\Delta y$, its centroid is $(\frac{1}{2}x, y)$, and its moment with respect to the x axis is $y(-x\,\Delta y)$. Then

$$M_x = -\int_0^3 yx\,dy = -\int_0^3 y(y^2-9)\,dy = \tfrac{81}{4}$$

Similarly, the moment of the approximating rectangle with respect to the y axis is $\frac{1}{2}x(-x\,\Delta y)$ and

$$M_y = -\tfrac{1}{2}\int_0^3 x^2\,dy = -\tfrac{1}{2}\int_0^3 (y^2-9)^2\,dy = -\tfrac{324}{5}$$

3. Determine the centroid of the first-quadrant area bounded by the parabola $y = 4 - x^2$.

The centroid of the approximating rectangle, shown in Fig. 43-3, is $(x, \frac{1}{2}y)$. Then its area is

$$A = \int_0^2 y\,dx = \int_0^2 (4-x^2)\,dx = \tfrac{16}{3}$$

and

$$M_x = \int_0^2 \tfrac{1}{2}y(y\,dx) = \tfrac{1}{2}\int_0^2 (4-x^2)^2\,dx = \tfrac{128}{15}$$

$$M_y = \int_0^2 xy\,dx = \int_0^2 x(4-x^2)\,dx = 4$$

Hence, $\bar{x} = M_y/A = \frac{3}{4}$, $\bar{y} = M_x/A = \frac{8}{5}$, and the centroid has coordinates $(\frac{3}{4}, \frac{8}{5})$.

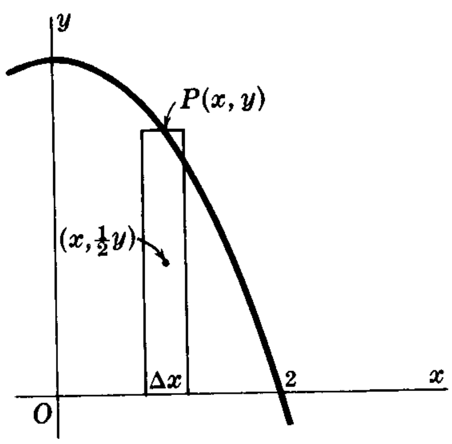

Fig. 43-3

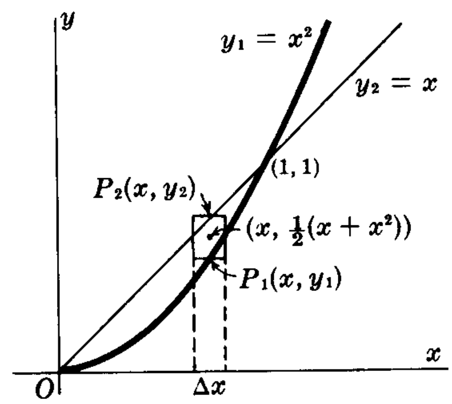

Fig. 43-4

4. Find the centroid of the first-quadrant area bounded by the parabola $y = x^2$ and the line $y = x$.

The centroid of the approximating rectangle, shown in Fig. 43-4, is $(x, \frac{1}{2}(x + x^2))$. Then

$$A = \int_0^1 (x - x^2)\,dx = \tfrac{1}{6}$$

$$M_x = \int_0^1 \tfrac{1}{2}(x+x^2)(x-x^2)\,dx = \tfrac{1}{15} \qquad M_y = \int_0^1 x(x-x^2)\,dx = \tfrac{1}{12}$$

Hence, $\bar{x} = M_y/A = \frac{1}{2}$, $\bar{y} = M_x/A = \frac{2}{5}$, and the coordinates of the centroid are $(\frac{1}{2}, \frac{2}{5})$.

5. Find the centroid of the area bounded by the parabolas $x = y^2$ and $x^2 = -8y$.

The centroid of the approximating rectangle, shown in Fig. 43-5, is $(x, \frac{1}{2}(-x^2/8 - \sqrt{x}))$. Then

$$A = \int_0^4 \left(-\frac{x^2}{8} + \sqrt{x}\right) dx = \frac{8}{3}$$

$$M_x = \int_0^4 \frac{1}{2}\left(-\frac{x^2}{8} - \sqrt{x}\right)\left(-\frac{x^2}{8} - \sqrt{x}\right) dx = -\frac{12}{5}$$

$$M_y = \int_0^4 x\left(-\frac{x^2}{8} + \sqrt{x}\right) dx = \frac{24}{5}$$

Hence the centroid is $(\bar{x}, \bar{y}) = (\frac{9}{5}, -\frac{9}{10})$.

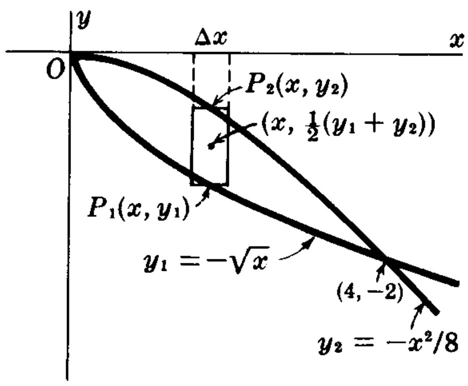

Fig. 43-5

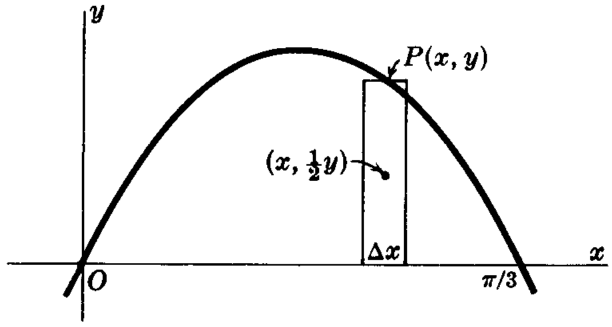

Fig. 43-6

6. Find the centroid of the area under the curve $y = 2\sin 3x$ from $x = 0$ to $x = \pi/3$.

The approximating rectangle, shown in Fig. 43-6, has the centroid $(x, \frac{1}{2}y)$. Then

$$A = \int_0^{\pi/3} y\,dx = \int_0^{\pi/3} 2\sin 3x\,dx = \left[-\frac{2}{3}\cos 3x\right]_0^{\pi/3} = \frac{4}{3}$$

$$M_x = \int_0^{\pi/3} \frac{1}{2}\,y(y\,dx) = 2\int_0^{\pi/3} \sin^2 3x\,dx = 2\left[\frac{1}{2}x - \frac{1}{12}\sin 6x\right]_0^{\pi/3} = \frac{\pi}{3}$$

$$M_y = \int_0^{\pi/3} xy\,dx = 2\int_0^{\pi/3} x\sin 3x\,dx = \frac{2}{9}\left[\sin 3x - 3x\cos 3x\right]_0^{\pi/3} = \frac{2}{9}\pi$$

The coordinates of the centroid are $(M_y/A, M_x/A) = (\pi/6, \pi/4)$.

7. Determine the centroid of the first-quadrant area of the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

By symmetry, $\bar{x} = \bar{y}$. (See Fig. 43-7.) We have

$$A = \int_{\theta=0}^{\theta=\pi/2} x\,dy = \int_0^{\pi/2} a\cos^3\theta(3a\sin^2\theta\cos\theta\,d\theta) = \frac{3}{4}a^2\int_0^{\pi/2}(\sin^2 2\theta)\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{3}{8}a^2\left[\frac{\theta}{2} - \frac{1}{8}\sin 4\theta + \frac{1}{6}\sin^3 2\theta\right]_0^{\pi/2} = \frac{3}{32}\pi a^2$$

$$M_x = \int_{\theta=0}^{\theta=\pi/2} yx\,dy = 3a^3\int_0^{\pi/2}\cos^4\theta\sin^5\theta\,d\theta = 3a^3\int_0^{\pi/2}\cos^4\theta(1-\cos^2\theta)^2\sin\theta\,d\theta$$

$$= -3a^3\left[\frac{\cos^5\theta}{5} - \frac{2\cos^7\theta}{7} + \frac{\cos^9\theta}{9}\right]_0^{\pi/2} = \frac{24a^3}{315}$$

Hence, $\bar{y} = M_x/A = 256a/315\pi$, and the centroid has coordinates $(256a/315\pi, 256a/315\pi)$.

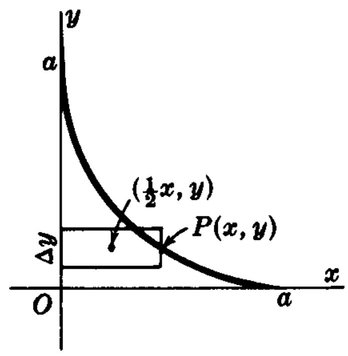

Fig. 43-7

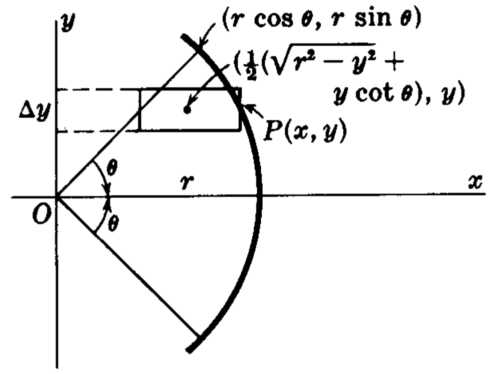

Fig. 43-8

8. Show that the centroid of a circular sector of radius r and angle 2θ is at a distance $\dfrac{2r \sin \theta}{3\theta}$ from the center of the circle.

Take the sector so that the centroid lies on the x axis (Fig. 43-8). By symmetry, the abscissa of the required centroid is that of the centroid of the area lying above the x axis bounded by the circle and the line $y = x \tan \theta$. For this latter sector,

$$A = \int_0^{r \sin \theta} (\sqrt{r^2 - y^2} - y \cot \theta)\, dy = \left[\frac{1}{2} y\sqrt{r^2 - y^2} + \frac{1}{2} r^2 \arcsin \frac{y}{r} - \frac{1}{2} y^2 \cot \theta\right]_0^{r \sin \theta} = \frac{1}{2} r^2\theta$$

$$M_y = \int_0^{r \sin \theta} \frac{1}{2} (\sqrt{r^2 - y^2} + y \cot \theta)(\sqrt{r^2 - y^2} - y \cot \theta)\, dy = \frac{1}{2} \int_0^{r \sin \theta} (r^2 - y^2 - y^2 \cot^2 \theta)\, dy$$

$$= \frac{1}{2}\left[r^2 y - \frac{1}{3} y^3 - \frac{1}{3} y^3 \cot^2 \theta\right]_0^{r \sin \theta} = \frac{1}{3} r^3 \sin \theta$$

$$\bar{x} = \frac{M_y}{A} = \frac{2r \sin \theta}{3\theta}$$

9. Find the centroid $(\bar{x}, 0)$ of the solid generated by revolving the area of Problem 3 about the x axis.

We use the approximating rectangle of Problem 3 and the disc method:

$$V = \pi \int_0^2 y^2\, dx = \pi \int_0^2 (4 - x^2)^2\, dx = \frac{256\pi}{15},$$

$$M_{yz} = \pi \int_0^2 xy^2\, dx = \pi \int_0^2 x(4 - x^2)^2\, dx = \frac{32\pi}{3}$$

and $\bar{x} = M_{yz}/V = \frac{5}{8}$.

10. Find the centroid $(0, \bar{y})$ of the solid generated by revolving the area of Problem 3 about the y axis.

We use the approximating rectangle of Problem 3 and the shell method:

$$V = 2\pi \int_0^2 xy\, dx = 2\pi \int_0^2 x(4 - x^2)\, dx = 8\pi$$

$$M_{xz} = 2\pi \int_0^2 \tfrac{1}{2} y(xy\, dx) = \pi \int_0^2 x(4 - x^2)^2\, dx = \frac{32\pi}{3}$$

and $\bar{y} = M_{xz}/V = \frac{4}{3}$.

11. Find the centroid $(\bar{x}, 0)$ of the solid generated by revolving the area of Problem 4 about the x axis.

We use the approximating rectangle of Problem 4 and the disc method:

$$V = \pi \int_0^1 (x^2 - x^4)\,dx = \frac{2\pi}{15} \qquad \text{and} \qquad M_{yz} = \pi \int_0^1 x(x^2 - x^4)\,dx = \frac{\pi}{12}$$

and $\bar{x} = M_{yz}/V = \frac{5}{8}$.

12. Find the centroid $(0, \bar{y})$ of the solid generated by revolving the area of Problem 4 about the y axis.

We use the approximating rectangle of Problem 4 and the shell method:

$$V = 2\pi \int_0^1 x(x - x^2)\,dx = \frac{\pi}{6} \qquad \text{and} \qquad M_{xz} = 2\pi \int_0^1 \frac{1}{2}(x + x^2)(x)(x - x^2)\,dx = \frac{\pi}{12}$$

and $\bar{y} = M_{xz}/V = \frac{1}{2}$.

13. Find the centroid of the area of a semicircle of radius r.

Take the semicircle as in Fig. 43-9, so that $\bar{x} = 0$. The area of the semicircle is $\frac{1}{2}\pi r^2$, the solid generated by revolving it about the x axis is a sphere of volume $\frac{4}{3}\pi r^3$, and the centroid $(0, \bar{y})$ of the area describes a circle of radius $\bar{y}$. Then, by the first theorem of Pappus, $\frac{1}{2}\pi r^2 \cdot 2\pi\bar{y} = \frac{4}{3}\pi r^3$, from which $\bar{y} = 4r/3\pi$. The centroid is at the point $(0, 4r/3\pi)$.

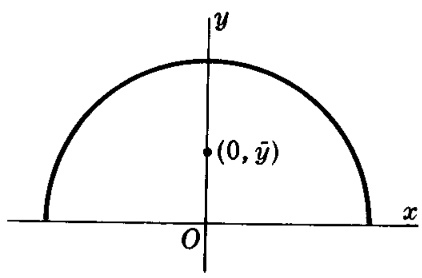

Fig. 43-9

14. Find the volume of the torus generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$. (See Fig. 43-10.)

The centroid of the disc describes a circle of radius 3. Hence, $V = \pi(2)^2 \cdot 2\pi(3) = 24\pi^2$ cubic units, by the first theorem of Pappus.

15. The rectangle of Fig. 43-11 is revolved about (a) the line $x = 9$, (b) the line $y = -5$, and (c) the line $y = -x$. Find the volume generated in each case.

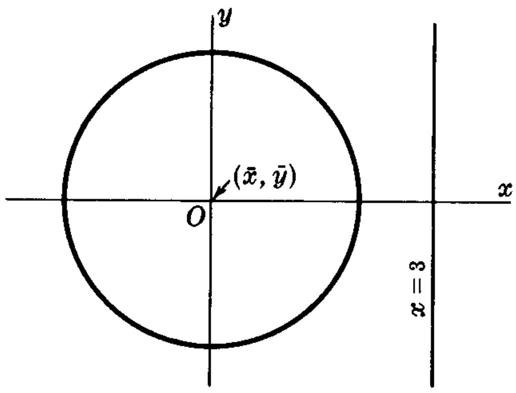

Fig. 43-10

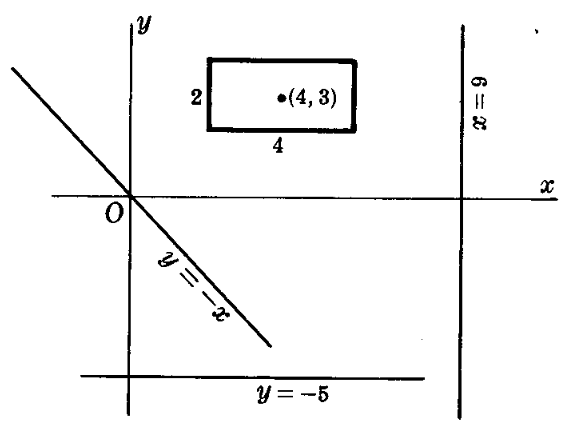

Fig. 43-11

(a) The centroid $(4, 3)$ of the rectangle describes a circle of radius 5. Hence, $V = 2(4)\cdot 2\pi(5) = 80\pi$ cubic units.
(b) The centroid describes a circle of radius 8. Hence, $V = 8(16\pi) = 128\pi$ cubic units.
(c) The centroid describes a circle of radius $(4+3)/\sqrt{2}$. Hence, $V = 56\sqrt{2}\pi$ cubic units.

Supplementary Problems

In Problems 16 to 26, find the centroid of the given area.

16. Between $y = x^2$, $y = 9$ *Ans.* $(0, \frac{27}{5})$

17. Between $y = 4x - x^2$, $y = 0$ *Ans.* $(2, \frac{8}{5})$

18. Between $y = 4x - x^2$, $y = x$ *Ans.* $(\frac{3}{2}, \frac{12}{5})$

19. Between $3y^2 = 4(3 - x)$, $x = 0$ *Ans.* $(\frac{6}{5}, 0)$

20. Within $x^2 = 8y$, $y = 0$, $x = 4$ *Ans.* $(3, \frac{3}{5})$

21. Between $y = x^2$, $4y = x^3$ *Ans.* $(\frac{12}{5}, \frac{192}{35})$

22. Between $x^2 - 8y + 4 = 0$, $x^2 = 4y$, in first quadrant *Ans.* $(\frac{3}{4}, \frac{2}{5})$

23. First-quadrant area of $x^2 + y^2 = a^2$ *Ans.* $(4a/3\pi, 4a/3\pi)$

24. First-quadrant area of $9x^2 + 16y^2 = 144$ *Ans.* $(16/3\pi, 4/\pi)$

25. Right loop of $y^2 = x^4(1 - x^2)$ *Ans.* $(32/15\pi, 0)$

26. First arch of $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ *Ans.* $(\pi, \frac{5}{6})$

27. Show that the distance of the centroid of a triangle from the base is one-third the altitude.

In Problems 28 to 38, find the centroid of the solid generated by revolving the given plane area about the given line.

28. Within $y = x^2$, $y = 9$, $x = 0$; about y axis *Ans.* $\bar{y} = 6$

29. Within $y = x^2$, $y = 9$, $x = 0$; about x axis *Ans.* $\bar{x} = \frac{5}{4}$

30. Within $y = 4x - x^2$, $y = x$; about x axis *Ans.* $\bar{x} = \frac{27}{16}$

31. Within $y = 4x - x^2$, $y = x$; about y axis *Ans.* $\bar{y} = \frac{27}{10}$

32. Within $x^2 - y^2 = 16$, $y = 0$, $x = 8$; about x axis *Ans.* $\bar{x} = \frac{27}{4}$

33. Within $x^2 - y^2 = 16$, $y = 0$, $x = 8$; about y axis *Ans.* $\bar{y} = 3\sqrt{3}/2$

34. Within $(x - 2)y^2 = 4$, $y = 0$, $x = 3$, $x = 5$; about x axis *Ans.* $\bar{x} = (2 + 2\ln 3)/(\ln 3)$

35. Within $x^2y = 16(4 - y)$, $x = 0$, $y = 0$, $x = 4$; about y axis *Ans.* $\bar{y} = 1/(\ln 2)$

36. First quadrant area bounded by $y^2 = 12x$ and its latus rectum; about x axis *Ans.* $\bar{x} = 2$

37. Area of Problem 36; about y axis *Ans.* $\bar{y} = \frac{5}{2}$

38. Area of Problem 36; about directrix *Ans.* $\bar{y} = \frac{75}{32}$

39. Prove the first theorem of Pappus.

40. Use the first theorem of Pappus to find (a) the volume of a right circular cone of altitude a and radius of base b; (b) the ring obtained by revolving the ellipse $4(x-6)^2 + 9(y-5)^2 = 36$ about the x axis.

Ans. (a) $\frac{1}{3}\pi ab^2$ cubic units; (b) $60\pi^2$ cubic units

41. For the area A bounded by $y = -x^2 - 3x + 6$ and $x + y - 3 = 0$, find (a) its centroid; (b) the volume generated when A is revolved about the bounding line.

Ans. (a) $(-1, 28/5)$; (b) $2\pi \dfrac{\bar{x} + \bar{y} - 3}{\sqrt{2}} A = \dfrac{256\sqrt{2}}{15}\pi$ cubic units

42. For the volume generated by revolving the area A (shaded in Fig. 43-12) about the bounding line L, obtain

$$V = 2\pi \frac{a\bar{x} + \bar{y} - b}{\sqrt{a^2+1}} A = \frac{2\pi}{\sqrt{a^2+1}}(aM_y + M_x - bA) = \frac{\pi}{\sqrt{a^2+1}} \int_r^s (y_C - y_L)^2\, dx$$

43. Use the formula of Problem 42 to obtain the volume generated by revolving the given area about the bounding line if
(a) $y = -x^2 - 3x + 6$ and L is $x + y - 3 = 0$
(b) $y = 2x^2$ and L is $2x - y + 4 = 0$

Ans. (a) see Problem 41; (b) $162\sqrt{5}\pi/25$ cubic units

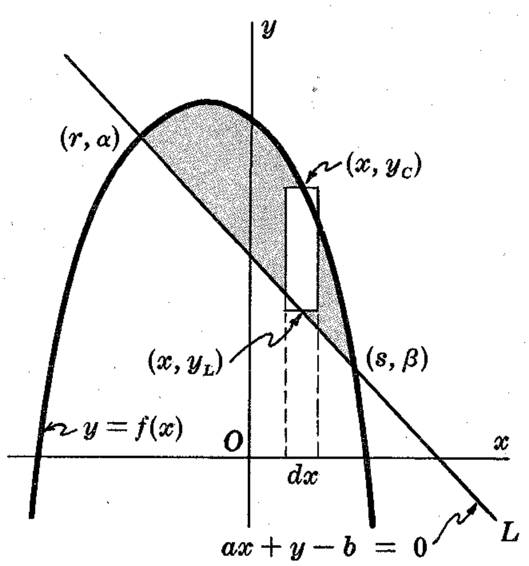

Fig. 43-12

Chapter 44

Moments of Inertia of Plane Areas and Solids of Revolution

THE MOMENT OF INERTIA I_L OF A PLANE AREA A with respect to a line L in its plane may be found as follows:

1. Make a sketch of the area, showing a representative strip parallel to the line and showing the approximating rectangle.
2. Form the product of the area of the rectangle and the square of the distance of its centroid from the line, and sum for all the rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

(See Problems 1 to 4.)

THE MOMENT OF INERTIA I_L OF A SOLID of volume V generated by revolving a plane area about a line L in its plane, with respect to line L, may be found as follows:

1. Make a sketch showing a representative strip parallel to the axis, and showing the approximating rectangle.
2. Form the product of the volume generated by revolving the rectangle about the axis (a shell) and the square of the distance of the centroid of the rectangle from the axis, and sum for all the rectangles.
3. Assume the number of rectangles to be indefinitely increased, and apply the fundamental theorem.

(See Problems 5 to 8.)

RADIUS OF GYRATION. The positive number R defined by the relation $I_L = AR^2$ in the case of a plane area A, and by $I_L = VR^2$ in the case of a solid of revolution, is called the *radius of gyration* of the area or volume with respect to L.

PARALLEL-AXIS THEOREM. The moment of inertia of an area, arc length, or volume with respect to any axis is equal to the moment of inertia with respect to a parallel axis through the centroid plus the product of the area, arc length, or volume and the square of the distance between the parallel axes. (See Problems 9 and 10.)

Solved Problems

1. Find the moment of inertia of a rectangular area A of dimensions a and b with respect to a side.

Take the rectangular area as in Fig. 44-1, and let the side in question be that along the y axis. The approximating rectangle has area $= b\,\Delta x$ and centroid $(x, \frac{1}{2}b)$. Hence its moment element is $x^2 b\,\Delta x$.

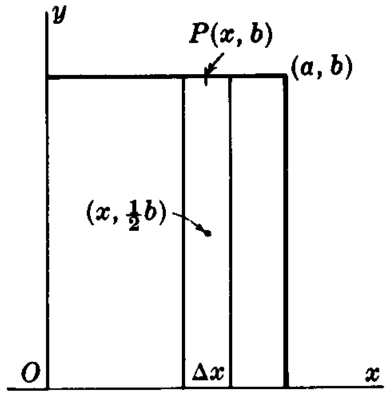

Fig. 44-1

Then

$$I_y = \int_0^a x^2 b\, dx = \left[b\,\frac{x^3}{3} \right]_0^a = \frac{ba^3}{3} = \frac{1}{3}\,Aa^2$$

Thus the moment of inertia of a rectangular area with respect to a side is one-third the product of the area and the square of the length of the other side.

2. Find the moment of inertia with respect to the y axis of the plane area between the parabola $y = 9 - x^2$ and the x axis. Also find the radius of gyration.

First solution: For the approximating rectangle of Fig. 44-2, $A = y\,\Delta x$ and the centroid is $(x, \frac{1}{2}y)$. Then

$$I_y = \int_{-3}^{3} x^2 y\, dx = 2\int_0^3 (9x^2 - x^4)\, dx = \tfrac{324}{5}$$

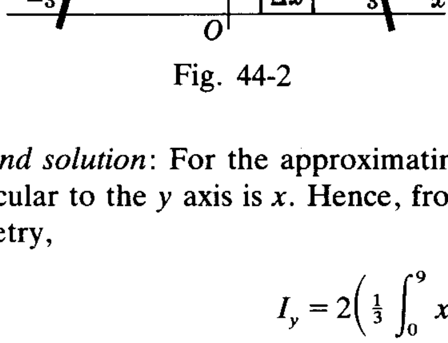

Fig. 44-2

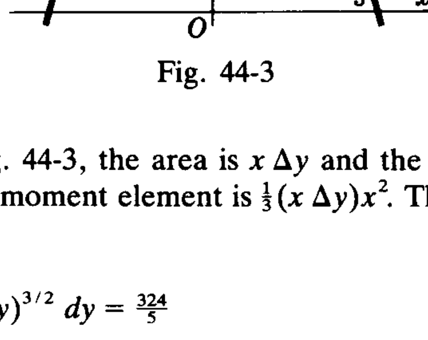

Fig. 44-3

Second solution: For the approximating rectangle of Fig. 44-3, the area is $x\,\Delta y$ and the dimension perpendicular to the y axis is x. Hence, from Problem 1, the moment element is $\frac{1}{3}(x\,\Delta y)x^2$. Thus, owing to symmetry,

$$I_y = 2\left(\tfrac{1}{3} \int_0^9 x^3\, dy \right) = \tfrac{2}{3} \int_0^9 (9 - y)^{3/2}\, dy = \tfrac{324}{5}$$

Since $I_y = \frac{324}{5} = AR^2$ and $A = 2\int_0^9 x\, dy = 2\int_0^9 \sqrt{9 - y}\, dy = 36$, the radius of gyration here is $R = 3/\sqrt{5}$.

3. Find the moment of inertia with respect to the y axis of the first-quadrant area bounded by the parabola $x^2 = 4y$ and the line $y = x$. Find the radius of gyration.

We use the approximating rectangle of Fig. 44-4, whose area is $(x - \frac{1}{4}x^2)\,\Delta x$ and whose centroid is

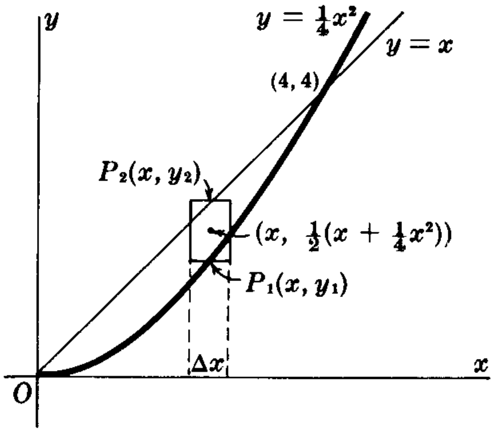

Fig. 44-4

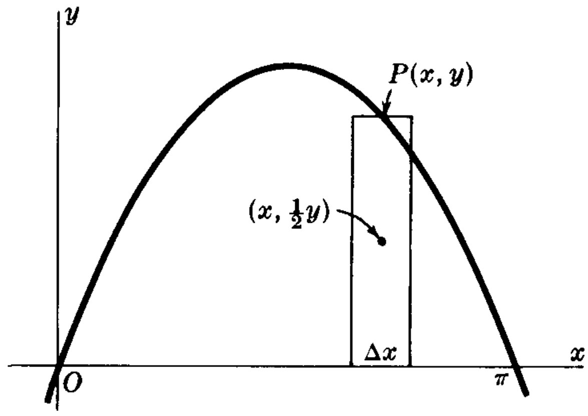

Fig. 44-5

$(x, \frac{1}{2}(x + \frac{1}{4}x^2))$. It yields

$$A = \int_0^4 (x - \tfrac{1}{4}x^2)\,dx = \tfrac{8}{3} \qquad I_y = \int_0^4 x^2(x - \tfrac{1}{4}x^2)\,dx = \tfrac{64}{5} = \tfrac{24}{5}A \qquad R = \sqrt{\tfrac{24}{5}} = \tfrac{2}{5}\sqrt{30}$$

4. Find the moment of inertia with respect to each coordinate axis of the area between the curve $y = \sin x$ from $x = 0$ to $x = \pi$ and the x axis.

From Fig. 44-5, $A = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = 2$ and

$$I_x = \int_0^\pi y^2(\tfrac{1}{3}\sin x\,dx) = \tfrac{1}{3}\int_0^\pi \sin^3 x\,dx = \tfrac{1}{3}[-\cos x + \tfrac{1}{3}\cos^3 x]_0^\pi = \tfrac{4}{9} = \tfrac{2}{9}A$$

$$I_y = \int_0^\pi x^2 \sin x\,dx = [2\cos x + 2x\sin x - x^2\cos x]_0^\pi = (\pi^2 - 4) = \tfrac{1}{2}(\pi^2 - 4)A$$

5. Find the moment of inertia with respect to its axis of a right circular cylinder whose height is b and whose base has radius a.

Let the cylinder be generated by revolving the rectangle of dimensions a and b about the y axis as in Fig. 44-6. For the approximating rectangle of the figure, the centroid is $(x, \frac{1}{2}b)$ and the volume of the shell generated by revolving the rectangle about the y axis is $\Delta V = 2\pi bx\,\Delta x$. Then, since $V = \pi ba^2$,

$$I_y = 2\pi\int_0^a x^2(bx\,dx) = \tfrac{1}{2}\pi ba^4 = \tfrac{1}{2}\pi ba^2 \cdot a^2 = \tfrac{1}{2}Va^2$$

Thus the moment of inertia with respect to its axis of a right circular cylinder is equal to one-half the product of its volume and the square of its radius.

6. Find the moment of inertia with respect to its axis of the solid generated by revolving about the x axis the area in the first quadrant bounded by the parabola $y^2 = 8x$, the x axis, and the line $x = 2$.

First solution: The centroid of the approximating rectangle of Fig. 44-7 is $(\frac{1}{2}(x + 2), y)$, and the volume generated by revolving the rectangle about the x axis is $2\pi y(2 - x)\,\Delta y = 2\pi y(2 - y^2/8)\,\Delta y$. Then

$$V = 2\pi\int_0^4 y\left(2 - \frac{y^2}{8}\right)dy = 16\pi \qquad \text{and} \qquad I_x = 2\pi\int_0^4 y^2\left[y\left(2 - \frac{y^2}{8}\right)dy\right] = \frac{256}{3}\pi = \frac{16}{3}V$$

Second solution: The volume generated by revolving the approximating rectangle of Fig. 44-8 about the x axis is $\pi y^2\,\Delta x$ and, by the result of Problem 5, its moment of inertia with respect to the x axis is $\frac{1}{2}y^2(\pi y^2\,\Delta x) = \frac{1}{2}\pi y^4\,\Delta x$. Then

$$V = \pi\int_0^2 y^2\,dx = 8\pi\int_0^2 x\,dx = 16\pi$$

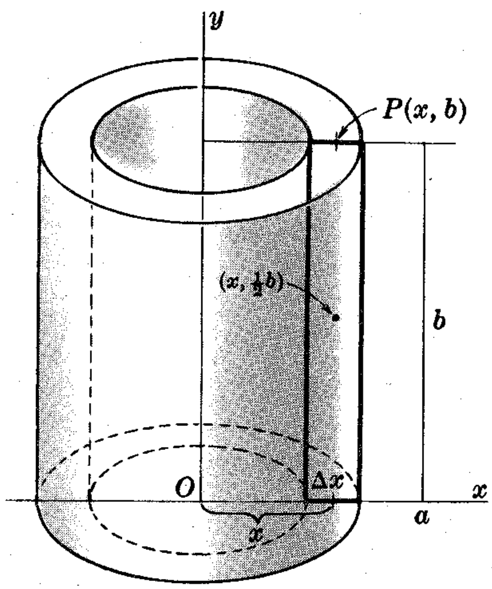

Fig. 44-6

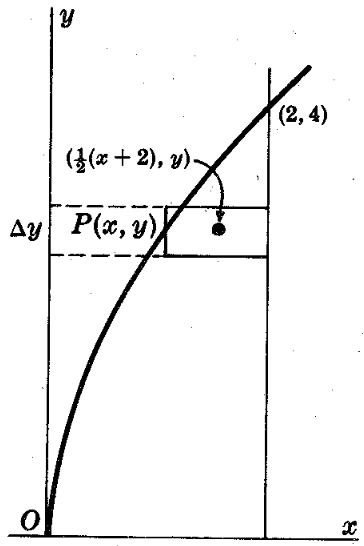

Fig. 44-7

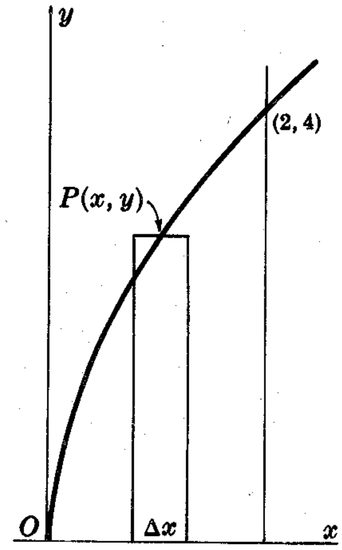

Fig. 44-8

and $$I_x = \tfrac{1}{2}\pi \int_0^2 y^4\,dx = 32\pi \int_0^2 x^2\,dx = \tfrac{256}{3}\pi = \tfrac{16}{3}V$$

7. Find the moment of inertia with respect to its axis of the solid generated by revolving the area of Problem 6 about the y axis.

The volume generated by revolving the approximating rectangle of Fig. 44-8 about the y axis is $2\pi xy\,\Delta x$. Then

$$V = 2\pi \int_0^2 xy\,dx = 4\sqrt{2}\pi \int_0^2 x^{3/2}\,dx = \tfrac{64}{5}\pi$$

and $$I_y = 2\pi \int_0^2 x^2(xy\,dx) = 4\sqrt{2}\pi \int_0^2 x^{7/2}\,dx = \tfrac{256}{9}\pi = \tfrac{20}{9}V$$

8. Find the moment of inertia with respect to its axis of the volume of the sphere generated by revolving a circle of radius r about a fixed diameter.

Take the circle as in Fig. 44-9, with the fixed diameter along the x axis. The shell method yields

$$V = 2\pi \int_0^r 2x(y\,dy) = \tfrac{4}{3}\pi r^3 \quad \text{and} \quad I_x = 4\pi \int_0^r y^2(xy\,dy) = 4\pi \int_0^r y^3\sqrt{r^2 - y^2}\,dy$$

Let $y = r\sin z$; then $\sqrt{r^2 - y^2} = r\cos z$ and $dy = r\cos z\,dz$. To change the y limits of integration to z limits, consider that when $y = 0$ then $0 = r\sin z$, so $0 = \sin z$ and $z = 0$; also, when $y = r$ then $r = r\sin z$, so $1 = \sin z$ and $z = \frac{1}{2}\pi$. Now

$$I_x = 4\pi r^5 \int_0^{\pi/2} \sin^3 z\cos^2 z\,dz = 4\pi r^5 \int_0^{\pi/2} (1 - \cos^2 z)\cos^2 z\sin z\,dz = \tfrac{8}{15}\pi r^5 = \tfrac{2}{5}r^2 V$$

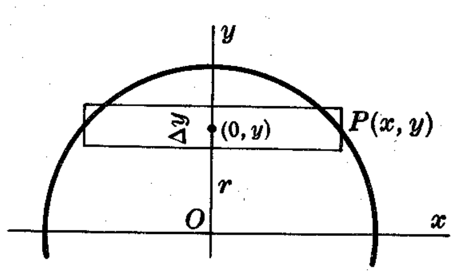

Fig. 44-9

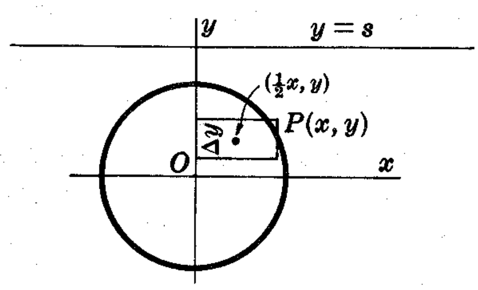

Fig. 44-10

9. Find the moment of inertia of the area of a circle of radius r with respect to a line s units from its center.

Take the center of the circle at the origin (see Fig. 44-10). We find first the moment of inertia of the circle with respect to the diameter parallel to the given line as

$$I_x = 4\int_0^r y^2(x\,dy) = 4\int_0^r y^2\sqrt{r^2-y^2}\,dy = \tfrac{1}{4}r^4\pi = \tfrac{1}{4}r^2A$$

Then $I_s = I_x + As^2 = (\frac{1}{4}r^2 + s^2)A$, by the parallel-axis theorem.

10. The moment of inertia with respect to its axis of the solid generated by revolving an arch of $y = \sin 3x$ about the x axis is $I_x = \pi^2/16 = 3V/8$. Find the moment of inertia of the solid with respect to the line $y = 2$.

By the parallel-axis theorem, $I_{y=2} = I_x + 2^2V = 3V/8 + 4V = 35V/8$.

Supplementary Problems

11. Find the moment of inertia of the given plane area with respect to the given line or lines.
(*a*) Within $y = 4 - x^2$, $x = 0$, $y = 0$; about x axis, y axis *Ans.* $128A/35$; $4A/5$
(*b*) Within $y = 8x^3$, $y = 0$, $x = 1$; about x axis, y axis *Ans.* $128A/15$; $2A/3$
(*c*) Within $x^2 + y^2 = a^2$; about a diameter *Ans.* $a^2A/4$
(*d*) Within $y^2 = 4x$, $x = 1$; about x axis, y axis *Ans.* $4A/5$; $3A/7$
(*e*) Within $4x^2 + 9y^2 = 36$; about x axis, y axis *Ans.* A; $9A/4$

12. Use the results of Problem 11 and the parallel-axis theorem to obtain the moment of inertia of the given area with respect to the given line: (*a*) within $y = 4 - x^2$, $y = 0$, $x = 0$, about $x = 4$; (*b*) within $x^2 + y^2 = a^2$, about a tangent; (*c*) within $y^2 = 4x$, $x = 1$, about $x = 1$.

Ans. (*a*) $84A/5$; (*b*) $5a^2A/4$; (*c*) $10A/7$

13. Find the moment of inertia with respect to its axis of the solid generated by revolving the given plane area about the given line:
(*a*) Within $y = 4x - x^2$, $y = 0$; about x axis, y axis (*b*) Within $y^2 = 8x$, $x = 2$; about x axis, y axis
(*c*) Within $4x^2 + 9y^2 = 36$; about x axis, y axis (*d*) Within $x^2 + y^2 = a^2$; about $y = b$, $b > a$

Ans. (*a*) $128V/21$, $32V/5$; (*b*) $16V/3$, $20V/9$; (*c*) $8V/5$, $18V/5$; (*d*) $(b^2 + \frac{3}{4}a^2)V$

14. Use the parallel-axis theorem to obtain the moment of inertia of: (*a*) a sphere of radius r about a line tangent to it; (*b*) a right circular cylinder about one of its elements. *Ans.* (*a*) $7r^2V/5$; (*b*) $3r^2V/2$

15. Prove: The moment of inertia of a plane area with respect to a line L perpendicular to its plane (or with respect to the foot of that perpendicular) is equal to the sum of its moments of inertia with respect to any two mutually perpendicular lines in the plane and through the foot of L.

16. Find the *polar moment of inertia* I_0 (the moment of inertia with respect to the origin) of: (*a*) the triangle bounded by $y = 2x$, $y = 0$, $x = 4$; (*b*) the circle of radius r and center at the origin; (*c*) the circle $x^2 - 2rx + y^2 = 0$; (*d*) the area bounded by the line $y = x$ and the parabola $y^2 = 2x$.

Ans. (*a*) $I_0 = I_x + I_y = 56A/3$; (*b*) $\frac{1}{2}r^2A$; (*c*) $3r^2A/2$; (*d*) $72A/35$

Chapter 45

Fluid Pressure

PRESSURE is defined as force per unit area:

$$p = \frac{\text{force acting perpendicular to an area}}{\text{area over which the force is distributed}}$$

The pressure p on a horizontal surface of area A due to a column of fluid of height h resting on it is $p = wh$, where w = weight of fluid per unit of volume. The force on this surface is $F = \text{pressure} \times \text{surface area} = whA$.

At any point within a fluid, the fluid exerts equal pressures in all directions.

FORCE ON A SUBMERGED PLANE AREA. Fig. 45-1 shows a plane area submerged vertically in a liquid of weight w pounds per unit of volume. Take the area to be in the xy plane, with the x axis in the surface of the liquid and the positive y axis directed downward. Divide the area into strips (always parallel to the surface of the liquid), and approximate each with a rectangle (as in Chapter 39).

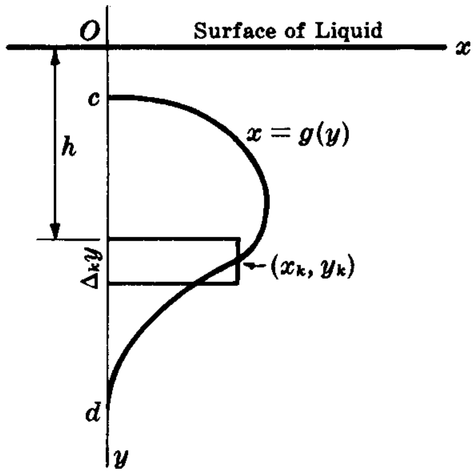

Fig. 45-1

Denote by h the depth of the upper edge of the representative rectangle of the figure. The force exerted on this rectangle of width $\Delta_k y$ and length $x_k = g(y_k)$ is $wY_k g(y_k)\,\Delta_k y$, where Y_k is some value of y between h and $h + \Delta_k y$. The total force on the plane area is, by the theorem of Bliss,

$$F = \lim_{n \to +\infty} \sum_{k=1}^{n} wY_k g(y_k)\,\Delta_k y = w \int_c^d y g(y)\,dy = w \int_c^d yx\,dy$$

Hence, the force exerted on a plane area submerged vertically in a liquid is equal to the product of the weight of a unit volume of the liquid, the submerged area, and the depth of the centroid of the area below the surface of the liquid. This, rather than a formula, should be used as the working principle in setting up such integrals.

Solved Problems

1. Find the force on one face of the rectangle submerged in water as shown in Fig. 45-2. Water weighs 62.5 lb/ft^3.

The submerged area is $2 \times 8 = 16$ ft^2, and its centroid is 1 ft below the water level. Hence,

$$F = \text{specific weight} \times \text{area} \times \text{depth of centroid} = 62.5 \text{ lb/ft}^3 \times 16 \text{ ft}^2 \times 1 \text{ ft} = 1000 \text{ lb}$$

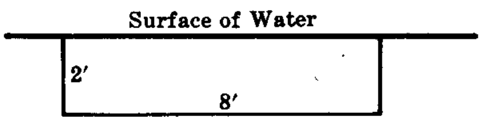

Fig. 45-2

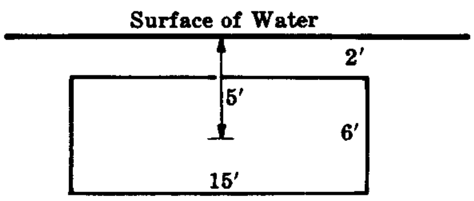

Fig. 45-3

2. Find the force on one face of the rectangle submerged in water as shown in Fig. 45-3.

The submerged area is 90 ft^2, and its centroid is 5 ft below the water level. Hence, $F = 62.5 \text{ lb/ft}^3 \times 90 \text{ ft}^2 \times 5 \text{ ft} = 28{,}125$ lb.

3. Find the force on one face of the triangle shown in Fig. 45-4. The units are feet, and the liquid weighs 50 lb/ft^3.

First solution: The submerged area is bounded by the lines $x = 0$, $y = 2$, and $3x + 2y = 10$. The force exerted on the approximating rectangle of area $x\,\Delta y$ and depth y is $wyx\,\Delta y = wy\,\dfrac{10-2y}{3}\,\Delta y$. Then $F = w\displaystyle\int_2^5 y\,\frac{10-2y}{3}\,dy = 9w = 450$ lb.

Second solution: The submerged area is 3 ft^2, and its centroid is $2 + \frac{1}{3}(3) = 3$ ft below the surface of the liquid. Hence, $F = 50(3)(3) = 450$ lb.

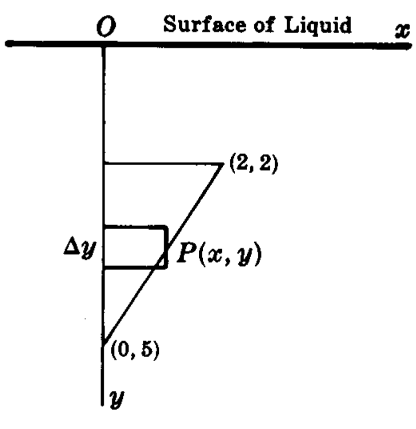

Fig. 45-4

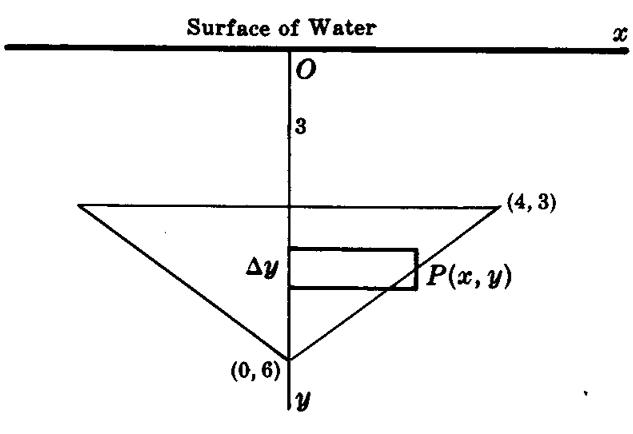

Fig. 45-5

4. A triangular plate whose edges are 5, 5, and 8 ft long is placed vertically in water with its longest edge uppermost, horizontal, and 3 ft below the water level. Calculate the force on a side of the plate.

First solution: Choosing the axes as in Fig. 45-5, we see that the required force is twice the force on the area bounded by the lines $y = 3$, $x = 0$, and $3x + 4y = 24$. The area of the approximating rectangle is $x\,\Delta y$, and its mean depth is y. Hence $\Delta F = wyx\,\Delta y = wy(8 - 4y/3)\,\Delta y$ and

$$F = 2w\int_3^6 y(8 - \tfrac{4}{3}y)\,dy = 48w = 3000 \text{ lb}$$

Second solution: The submerged area is 12 ft^2, and its centroid is $3 + \frac{1}{3}(3) = 4$ ft below the water level. Hence $F = 62.5(12)(4) = 3000$ lb.

5. Find the force on the end of a trough in the form of a semicircle of radius 2 ft, when the trough is filled with a liquid weighing 60 lb/ft^3.

With the coordinate system chosen as in Fig. 45-6, the force on the approximating rectangle is $wyx\,\Delta y = wy\sqrt{4-y^2}\,\Delta y$. Hence $F = 2w\int_0^2 y\sqrt{4-y^2}\,dy = \frac{16}{3}w = 320$ lb.

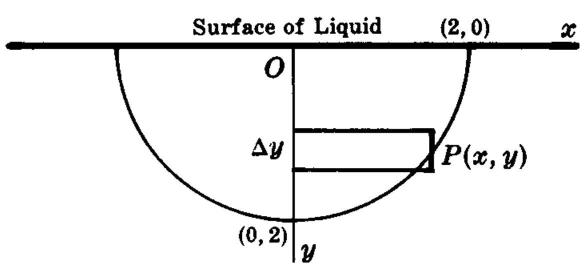

Fig. 45-6

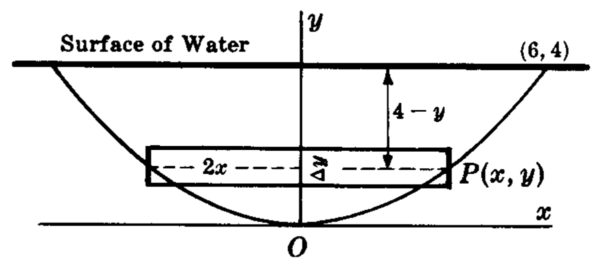

Fig. 45-7

6. A plate in the form of a parabolic segment of base 12 ft and height 4 ft is submerged in water so that its base is at the surface of the liquid. Find the force on a face of the plate.

With the coordinate system chosen as in Fig. 45-7, the equation of the parabola is $x^2 = 9y$. The area of the approximating rectangle is $2x\,\Delta y$, and the mean depth is $4 - y$. Then

$$\Delta F = 2w(4-y)x\,\Delta y = 2w(4-y)(3\sqrt{y}\,\Delta y) \qquad \text{and} \qquad F = 6w\int_0^4 (4-y)\sqrt{y}\,dy = \frac{256}{5}w = 3200 \text{ lb}$$

7. Find the force on the plate of Problem 6 if it is partly submerged in a liquid weighing 48 lb/ft^3 so that its axis is parallel to and 3 ft below the surface of the liquid.

With the coordinate system chosen as in Fig. 45-8, the equation of the parabola is $y^2 = 9x$. The area of the approximating rectangle is $(4-x)\,\Delta y$, its mean depth is $3-y$, and the force on it is $\Delta F = w(3-y)(4-x)\,\Delta y = w(3-y)(4-y^2/9)\,\Delta y$. Then

$$F = w\int_{-6}^{3} (3-y)\left(4 - \frac{y^2}{9}\right) dy = \frac{405}{4}\,w = 4860 \text{ lb}$$

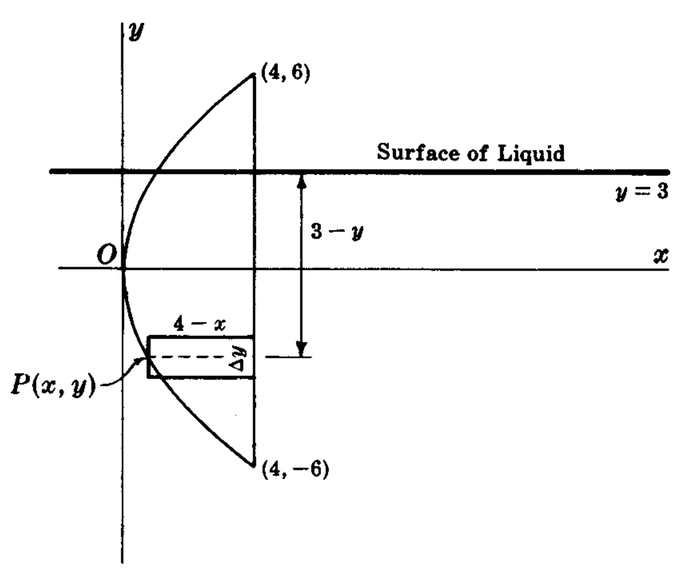

Fig. 45-8

Supplementary Problems

8. A 6-ft by 8-ft rectangular plate is submerged vertically in a liquid weighing w lb/ft^3. Find the force on one face
(*a*) If the shorter side is uppermost and lies in the surface of the liquid
(*b*) If the shorter side is uppermost and lies 2 ft below the surface of the liquid
(*c*) If the longer side is uppermost and lies in the surface of the liquid
(*d*) If the plate is held by a rope attached to a corner 2 ft below the liquid surface

Ans. (*a*) $192w$ lb; (*b*) $288w$ lb; (*c*) $144w$ lb; (*d*) $336w$ lb

9. Assuming the x axis horizontal and the positive y axis directed downward, find the force on a side of each of the following areas. The dimensions are in feet, and the fluid weighs w lb/ft^3.
(*a*) Within $y = x^2$, $y = 4$; fluid surface at $y = 0$ *Ans.* $128w/5$ lb
(*b*) Within $y = x^2$, $y = 4$; fluid surface at $y = -2$ *Ans.* $704w/15$ lb
(*c*) Within $y = 4 - x^2$, $y = 0$; fluid surface at $y = 0$ *Ans.* $256w/15$ lb
(*d*) Within $y = 4 - x^2$, $y = 0$; fluid surface at $y = -3$ *Ans.* $736w/15$ lb
(*e*) Within $y = 4 - x^2$, $y = 2$; fluid surface at $y = -1$ *Ans.* $152\sqrt{2}w/15$ lb

10. A trough of trapezoidal cross section is 2 ft wide at the bottom, 4 ft wide at the top, and 3 ft deep. Find the force on an end (*a*) if it is full of water; (*b*) if it contains 2 ft of water.

Ans. (*a*) 750 lb; (*b*) 305.6 lb

11. A circular plate of radius 2 ft is lowered into a liquid weighing w lb/ft^3 so that its center is 4 ft below the surface. Find the force on the lower half of the plate and on the upper half.

Ans. $(8\pi + 16/3)w$ lb; $(8\pi - 16/3)w$ lb

12. A cylindrical tank 6 ft in radius is lying on its side. If it contains oil weighing w lb/ft^3 to a depth of 9 ft, find the force on an end. *Ans.* $(72\pi + 81\sqrt{3})w$ lb

13. The *center of pressure* of the area of Fig. 45-1 is that point $(\bar{x}, \bar{y})$ where a concentrated force of magnitude F would yield the same moment with respect to any horizontal or vertical line as the distributed forces.
(*a*) Show that $F\bar{x} = \frac{1}{2}w\int_c^d yx^2\,dy$ and $F\bar{y} = w\int_c^d y^2x\,dy$.
(*b*) Show that the depth of the center of pressure below the surface of the liquid is equal to the moment of inertia of the area divided by the first moment of the area, each with respect to a line in the surface of the liquid.

14. Use part (*b*) of Problem 13 to find the depth of the center of pressure below the surface of the liquid in (*a*) Problem 5; (*b*) Problem 6; (*c*) Problem 7; (*d*) Problem 9(*a*); (*e*) Problem 9(*b*).

Ans. (*a*) $3\pi/8$; (*b*) $\frac{16}{7}$; (*c*) $\frac{126}{25}$; (*d*) $\frac{20}{7}$; (*e*) $\frac{358}{77}$

Chapter 46

Work

CONSTANT FORCE. The work W done by a constant force F acting over a directed distance s along a straight line is Fs units.

VARIABLE FORCE. Consider a continuously varying force acting along a straight line. Let x denote the directed distance of the point of application of the force from a fixed point on the line, and let the force be given as some function $F(x)$ of x. To find the work done as the point of application moves from $x = a$ to $x = b$ (Fig. 46-1):

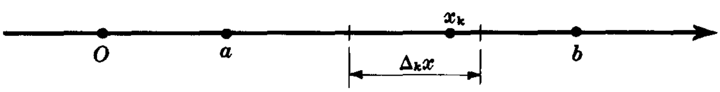

Fig. 46-1

1. Divide the interval $a \leqq x \leqq b$ into n subintervals of length $\Delta_k x$, and let x_k be any point in the kth subinterval.
2. Assume that during the displacement over the kth subinterval the force is constant and equal to $F(x_k)$. The work done during this displacement is then $F(x_k)\,\Delta_k x$, and the total work done by the set of n assumed constant forces is given by $\sum_{k=1}^{n} F(x_k)\,\Delta_k x$.
3. Increase the number of subintervals indefinitely in such a manner that each $\Delta_k x \to 0$ and apply the fundamental theorem to obtain

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k)\,\Delta_k x = \int_a^b F(x)\,dx$$

Solved Problems

1. Within certain limits, the force required to stretch a spring is proportional to the stretch, the constant of proportionality being called the *modulus* of the spring. If a given spring at its normal length of 10 inches requires a force of 25 lb to stretch it $\frac{1}{4}$ inch, calculate the work done in stretching it from 11 to 12 inches.

Let x denote the stretch; then $F(x) = kx$. When $x = \frac{1}{4}$, $F(x) = 25$; hence $25 = \frac{1}{4}k$, so that $k = 100$ and $F(x) = 100x$.

The work corresponding to a stretch Δx is $100x\,\Delta x$, and the required work is $W = \int_1^2 100x\,dx = 150$ in-lb.

2. The modulus of the spring on a bumping post in a freight yard is 270,000 lb/ft. Find the work done in compressing the spring 1 inches.

Let x be the displacement of the free end of the spring in feet. Then $F(x) = 270{,}000x$, and the work corresponding to a displacement Δx is $270{,}000x\,\Delta x$. Hence, $W = \int_0^{1/12} 270{,}000x\,dx = 937.5$ ft-lb.

3. A cable weighing 3 lb/ft is unwinding from a cylindrical drum. If 50 ft are already unwound, find the work done by the force of gravity as an additional 250 ft are unwound.

Let x = length of cable unwound at any time. Then $F(x) = 3x$ and $W = \int_{50}^{300} 3x\,dx = 131{,}250$ ft-lb.

4. A 100-ft cable weighing 5 lb/ft supports a safe weighing 500 lb. Find the work done in winding 80 ft of the cable on a drum.

Let x denote the length of cable that has been wound on the drum. The total weight (unwound cable and safe) is $500 + 5(100 - x) = 1000 - 5x$, and the work done in raising the safe a distance Δx is $(1000 - 5x)\,\Delta x$. Thus, the required work is $W = \int_0^{80} (1000 - 5x)\,dx = 64{,}000$ ft-lb.

5. A right circular cylindrical tank of radius 2 ft and height 8 ft is full of water. Find the work done in pumping the water to the top of the tank. Assume that the water weighs 62.5 lb/ft^3.

First solution: Imagine the water being pushed up by means of a piston that moves upward from the bottom of the tank. Figure 46-2 shows the piston when it is y ft from the bottom. The lifting force, being equal to the weight of the water remaining on the piston, is approximately $F(y) = \pi r^2 w(8 - y) = 4\pi w(8 - y)$, and the work corresponding to a displacement Δy of the piston is approximately $4\pi w(8 - y)\,\Delta y$. The work done in emptying the tank is then

$$W = 4\pi w \int_0^8 (8 - y)\,dy = 128\pi w = 128\pi(62.5) = 8000\pi \text{ ft lb}$$

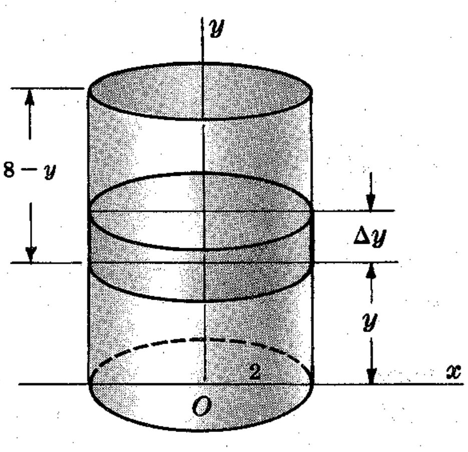

Fig. 46-2

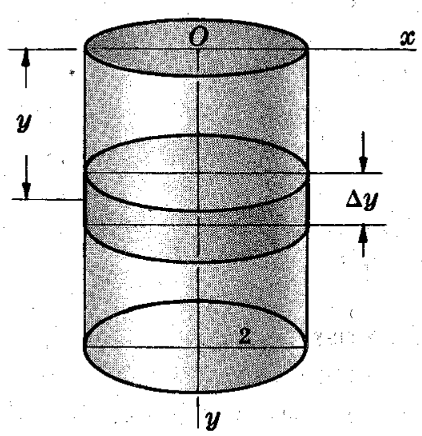

Fig. 46-3

Second solution: Imagine that the water in the tank is sliced into n disks of thickness Δy, and that the tank is to be emptied by lifting each disk to the top. For the representative disk of Fig. 46-3, whose mean distance from the top is y, the weight is $4\pi w\,\Delta y$ and the work done in moving it to the top of the tank is $4\pi wy\,\Delta y$. Summing for the n disks and applying the fundamental theorem, we have $W = 4\pi w \int_0^8 y\,dy = 128\pi w = 8000\pi$ ft-lb.

6. The expansion of a gas in a cylinder causes a piston to move so that the volume of the enclosed gas increases from 15 to 25 in^3. Assuming the relation between the pressure (p lb/in^2) and the volume (v in^3) to be $pv^{1.4} = 60$, find the work done.

If A denotes the area of a cross section of the cylinder, pA is the force exerted by the gas. A volume increase Δv causes the piston to move a distance $\Delta v/A$, and the work corresponding to this displacement is $pA\,\frac{\Delta v}{A} = \frac{60}{v^{1.4}}\,\Delta v$. Then,

$$W = 60\int_{15}^{25} \frac{dv}{v^{1.4}} = \left[-\frac{60}{0.4}\,v^{-0.4}\right]_{15}^{25} = -150\left(\frac{1}{25^{0.4}} - \frac{1}{15^{0.4}}\right) = 9.39 \text{ in-lb}$$

7. A conical vessel is 12 ft across the top and 15 ft tall. If it contains a liquid weighing w lb/ft^3 to a depth of 10 ft, find the work done in pumping the liquid to a height 3 ft above the top of the vessel.

Consider the representative disk in Fig. 46-4 whose radius is x, thickness is Δy, and mean distance from the bottom of the vessel is y. Its weight is $\pi w x^2 \, \Delta y$, and the work done in lifting it to the required height is $\pi w x^2(18 - y)\, \Delta y$.

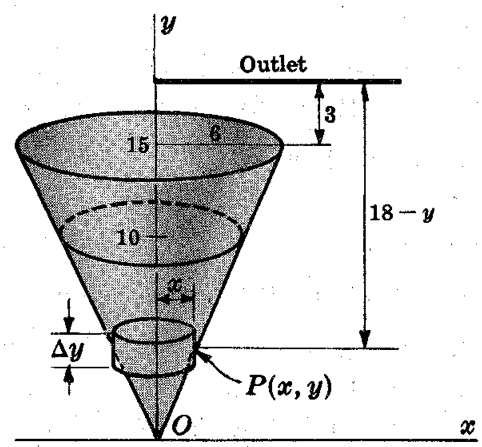

Fig. 46-4

From similar triangles, $\dfrac{x}{y} = \dfrac{6}{15}$; so $x = \dfrac{2}{5}\, y$. Then $W = \dfrac{4}{25}\, \pi w \displaystyle\int_0^{10} y^2(18 - y)\, dy = 560\pi w$ ft-lb.

Supplementary Problems

8. If a force of 80 lb stretches a 12-ft spring 1 ft, find the work done in stretching it (*a*) from 12 to 15 ft; (*b*) from 15 to 16 ft. *Ans.* (*a*) 360 ft-lb; (*b*) 280 ft-lb

9. Two particles repel each other with a force that is inversely proportional to the square of the distance between them. If one particle remains fixed at a point on the x axis 2 units to the right of the origin, find the work done in moving the second along the x axis to the origin from a point 3 units to the left of the origin. *Ans.* $3k/10$

10. The force with which the earth attracts a weight of w pounds at a distance s miles from its center is $F = (4000)^2 w/s^2$, where the radius of the earth is taken as 4000 mi. Find the work done against the force of gravity in moving a 1-lb mass from the surface of the earth to a point 1000 mi above the surface. *Ans.* 800 mi-lb

11. Find the work done against the force of gravity in moving a rocket weighing 8 tons to a height 200 mi above the surface of the earth. *Ans.* 32,000/21 mi-tons

12. Find the work done in lifting 1000 lb of coal from a mine 1500 ft deep by means of a cable weighing 2 lb/ft. *Ans.* 1875 ft-tons

13. A cistern is 10 ft square and 8 ft deep. Find the work done in emptying it over the top if (*a*) it is full of water; (*b*) it is three-quarters full of water. *Ans.* (*a*) 200,000 ft-lb; (*b*) 187,500 ft-lb

14. A hemispherical tank of radius 3 ft is full of water. (*a*) Find the work done in pumping the water over the edge of the tank. (*b*) Find the work done in emptying the tank through an outlet pipe 2 ft above the top of the tank. *Ans.* (*a*) 3976 ft-lb; (*b*) 11,045 ft-lb

15. How much work is done in filling an upright cylindrical tank of radius 3 ft and height 10 ft with liquid weighing w lb/ft^3 through a hole in the bottom? How much if the tank is horizontal?

Ans. $450\pi w$ ft-lb; $270\pi w$ ft-lb

16. Show that the work done in pumping out a tank is equal to the work that would be done by lifting the contents from the center of gravity of the liquid to the outlet.

17. A 200-lb weight is to be dragged 60 ft up a 30° ramp. If the force of friction opposing the motion is $N\mu$, where $\mu = 1/\sqrt{3}$ is the coefficient of friction and $N = 200\cos 30°$ is the normal force between weight and ramp, find the work done. *Ans.* 12,000 ft-lb

18. Solve Problem 17 for a 45° ramp with the coefficient of friction $\mu = 1/\sqrt{2}$. *Ans.* $6000(1 + \sqrt{2})$ ft-lb

19. Air is confined in a cylinder fitted with a piston. At a pressure of 20 lb/ft^2, the volume is 100 ft^3. Find the work done on the piston when the air is compressed to 2 ft^3 (*a*) assuming $pv = \text{constant}$; (*b*) assuming $pv^{1.4} = \text{constant}$. *Ans.* (*a*) 7824 ft-lb; (*b*) 18,910 ft-lb

Chapter 47

Length of Arc

THE LENGTH OF AN ARC AB of a curve is by definition the limit of the sum of the lengths of a set of consecutive chords AP_1, $P_1P_2, \ldots, P_{n-1}B$, joining points on the arc, when the number of points is indefinitely increased in such a manner that the length of each chord approaches zero (Fig. 47-1).

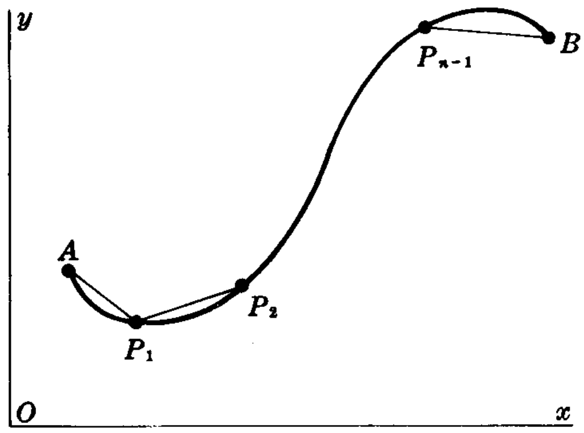

Fig. 47-1

If $A(a, c)$ and $B(b, d)$ are two points on the curve $y = f(x)$, where $f(x)$ and its derivative $f'(x)$ are continuous on the interval $a \le x \le b$, the length of arc AB is given by

$$s = \int_{AB} ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Similarly, if $A(a, c)$ and $B(b, d)$ are two points on the curve $x = g(y)$, where $g(y)$ and its derivative with respect to y are continuous on the interval $c \le y \le d$, the length of arc AB is given by

$$s = \int_{AB} ds = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

If $A(u = u_1)$ and $B(u = u_2)$ are two points on a curve defined by the parametric equations $x = f(u)$, $y = g(u)$, and if conditions of continuity are satisfied, the length of arc AB is given by

$$s = \int_{AB} ds = \int_{u_1}^{u_2} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

(For a derivation, see Problem 1.)

Solved Problems

1. Derive the arc-length formula $s = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx$.

Let the interval $a \le x \le b$ be divided into subintervals by the insertion of points $\xi_0 = a$, ξ_1, $\xi_2, \ldots, \xi_{n-1}$, $\xi_n = b$, and erect perpendiculars to determine the points $P_0 = A$, P_1, $P_2, \ldots, P_{n-1}$,

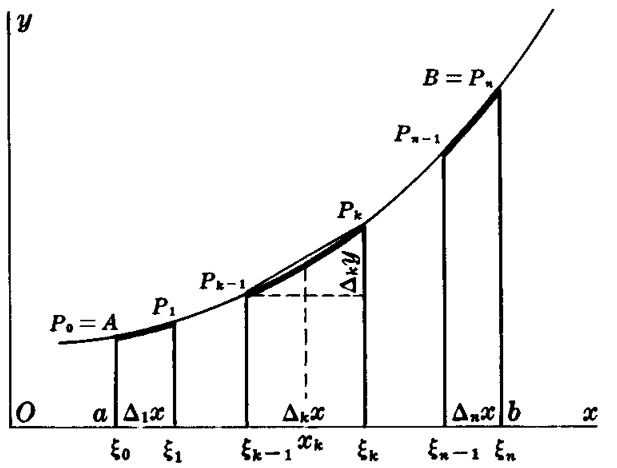

Fig. 47-2

$P_n = B$ on the arc as in Fig. 47-2. For the representative chord of the figure,

$$P_{k-1}P_k = \sqrt{(\Delta_k x)^2 + (\Delta_k y)^2} = \sqrt{1 + \left(\frac{\Delta_k y}{\Delta_k x}\right)^2}\ \Delta_k x$$

By the law of the mean (Chapter 26), there is at least one point, say $x = x_k$, on the arc $P_{k-1}P_k$ where the slope of the tangent $f'(x_k)$ is equal to the slope $\Delta_k y/\Delta_k x$ of the chord $P_{k-1}P_k$. Thus,

$$P_{k-1}P_k = \sqrt{1 + [f'(x_k)]^2}\ \Delta_k x \qquad \text{for } \xi_{k-1} < x_k < \xi_k$$

and, using the fundamental theorem, we have

$$AB = \lim_{n\to+\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k)]^2}\ \Delta_k x = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

2. Find the length of the arc of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.

Since $dy/dx = \frac{3}{2}x^{1/2}$,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_0^5 \sqrt{1 + \frac{9}{4}x}\ dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^5 = \frac{335}{27} \text{ units}$$

3. Find the length of the arc of the curve $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.

Since $dx/dy = \frac{9}{2}y^{1/2}$,

$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy = \int_0^4 \sqrt{1 + \frac{81}{4}y}\ dy = \frac{8}{243}(82\sqrt{82} - 1) \text{ units}$$

4. Find the length of the arc of $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

$\dfrac{dy}{dx} = \dfrac{x^4 - 16}{8x^2}$ and $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{1}{64}\left(\dfrac{x^4 + 16}{x^2}\right)^2$. Then $s = \dfrac{1}{8}\displaystyle\int_2^4 \left(x^2 + \frac{16}{x^2}\right) dx = \frac{17}{6}$ units.

5. Find the length of the arc of the catenary $y = \frac{1}{2}a(e^{x/a} + e^{-x/a})$ from $x = 0$ to $x = a$.

$\dfrac{dy}{dx} = \dfrac{1}{2}(e^{x/a} - e^{-x/a})$ and $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1}{4}(e^{2x/a} - 2 + e^{-2x/a}) = \dfrac{1}{4}(e^{x/a} + e^{-x/a})^2$. Then

$$s = \frac{1}{2}\int_0^a (e^{x/a} + e^{-x/a})\ dx = \frac{1}{2}a[e^{x/a} - e^{-x/a}]_0^a = \frac{1}{2}a\left(e - \frac{1}{e}\right) \text{ units}$$

6. Find the length of the arc of the parabola $y^2 = 12x$ cut off by its latus rectum.

The required length is twice that from the point $(0,0)$ to the point $(3,6)$. We have $\dfrac{dx}{dy} = \dfrac{y}{6}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{36+y^2}{36}$. Then

$$s = 2(\tfrac{1}{6}) \int_0^6 \sqrt{36+y^2}\, dy = \tfrac{1}{3}[\tfrac{1}{2}y\sqrt{36+y^2} + 18 \ln (y + \sqrt{36+y^2})]_0^6$$
$$= 6[\sqrt{2} + \ln (1+\sqrt{2})] \text{ units}$$

7. Find the length of the arc of the curve $x = t^2$, $y = t^3$ from $t = 0$ to $t = 4$.

Here $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$, and $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = 4t^2 + 9t^4 = 4t^2\left(1 + \dfrac{9}{4}t^2\right)$. Then

$$s = \int_0^4 \sqrt{1 + \frac{9}{4}t^2}\,(2t\,dt) = \frac{8}{27}(37\sqrt{37} - 1) \text{ units}$$

8. Find the length of an arch of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$.

An arch is described as θ varies from $\theta = 0$ to $\theta = 2\pi$. We have $\dfrac{dx}{d\theta} = 1 - \cos\theta$, $\dfrac{dy}{d\theta} = \sin\theta$, and $\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2 = 2(1 - \cos\theta) = 4\sin^2 \tfrac{1}{2}\theta$. Then $s = 2\displaystyle\int_0^{2\pi} \sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 8$ units.

Supplementary Problems

In Problems 9 to 20, find the length of the entire curve or indicated arc.

9. $y^3 = 8x^2$ from $x = 1$ to $x = 8$ *Ans.* $(104\sqrt{13} - 125)/27$ units

10. $6xy = x^4 + 3$ from $x = 1$ to $x = 2$ *Ans.* $\frac{17}{12}$ units

11. $y = \ln x$ from $x = 1$ to $x = 2\sqrt{2}$ *Ans.* $3 - \sqrt{2} + \ln \frac{1}{2}(2+\sqrt{2})$ units

12. $27y^2 = 4(x-2)^3$ from $(2,0)$ to $(11, 6\sqrt{3})$ *Ans.* 14 units

13. $y = \ln (e^x - 1)/e^x + 1$ from $x = 2$ to $x = 4$ *Ans.* $\ln (e^4 + 1) - 2$ units

14. $y = \ln (1 - x^2)$ from $x = \frac{1}{4}$ to $x = \frac{3}{4}$ *Ans.* $\ln \frac{21}{5} - \frac{1}{2}$ units

15. $y = \frac{1}{2}x^2 - \frac{1}{4}\ln x$ from $x = 1$ to $x = e$ *Ans.* $\frac{1}{2}e^2 - \frac{1}{4}$ units

16. $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$ *Ans.* $\ln (1+\sqrt{2})/\sqrt{3}$ units

17. $x = a\cos\theta$, $y = a\sin\theta$ *Ans.* $2\pi a$ units

18. $x = e^t \cos t$, $y = e^t \sin t$ from $t = 0$ to $t = 4$ *Ans.* $\sqrt{2}(e^4 - 1)$ units

19. $x = \ln\sqrt{1+t^2}$, $y = \arctan t$ from $t = 0$ to $t = 1$ *Ans.* $\ln (1+\sqrt{2})$ units

20. $x = 2\cos\theta + \cos 2\theta + 1$, $y = 2\sin\theta + \sin 2\theta$ *Ans.* 16 units

21. The position of a point at time t is given as $x = \frac{1}{2}t^2$, $y = \frac{1}{9}(6t+9)^{3/2}$. Find the distance the point travels from $t = 0$ to $t = 4$. *Ans.* 20 units

22. Let $P(x, y)$ be a fixed point and $Q(x + \Delta x, y + \Delta y)$ be a variable point on the curve $y = f(x)$. (See Fig. 22-1.) Show that

$$\lim_{Q\to P} \frac{\text{arc } PQ}{\text{chord } PQ} = \lim_{Q\to P} \frac{\Delta s}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = \frac{ds/dx}{\sqrt{1 + (dy/dx)^2}} = 1$$

23. (*a*) Show that the length of the first-quadrant arc of $x = a\cos^3\theta$, $y = a\sin^3\theta$ is $3a/2$.

(*b*) Show that when the arc length of (*a*) is computed from $x^{2/3} + y^{2/3} = a^{2/3}$ we obtain $a^{1/3}\int_0^a \frac{dx}{x^{1/3}}$, in which the integrand is infinite at the lower limit of integration. Definite integrals of this type will be considered in Chapter 52.

24. A problem leading to the so-called *curve of pursuit* may be formulated as follows: A dog at $A(1, 0)$ sees his master at $(0, 0)$ walking along the y axis and runs (in the first quadrant) to meet him. Find the path of the dog assuming that it is always headed toward its master and that each moves at a constant rate, p for the master and $q > p$ for the dog. This problem can be solved in Chapter 76. Verify here that the equation $y = f(x)$ of the path may be found by integrating $y' = \frac{1}{2}(x^{p/q} - x^{-p/q})$.

(*Hint*: Let $P(a, b)$, for $0 < a < 1$, be a position of the dog, and denote by Q the intersection of the y axis and the tangent to $y = f(x)$ at P. Find the time required for the dog to reach P, and show that the master is then at Q.)

Chapter 48

Area of a Surface of Revolution

THE AREA OF THE SURFACE generated by revolving the arc AB of a continuous curve about a line in its plane is by definition the limit of the sum of the areas generated by the n consecutive chords $AP_1, P_1P_2, \ldots, P_{n-1}B$ joining points on the arc when revolved about the line, as the number of chords is indefinitely increased in such a manner that the length of each chord approaches zero.

If $A(a, c)$ and $B(b, d)$ are two points of the curve $y = f(x)$, where $f(x)$ and $f'(x)$ are continuous and $f(x)$ does not change sign on the interval $a \le x \le b$ (Fig. 48-1), the area of the surface generated by revolving the arc AB about the x axis is given by

$$S_x = 2\pi \int_{AB} y\, ds = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{48.1}$$

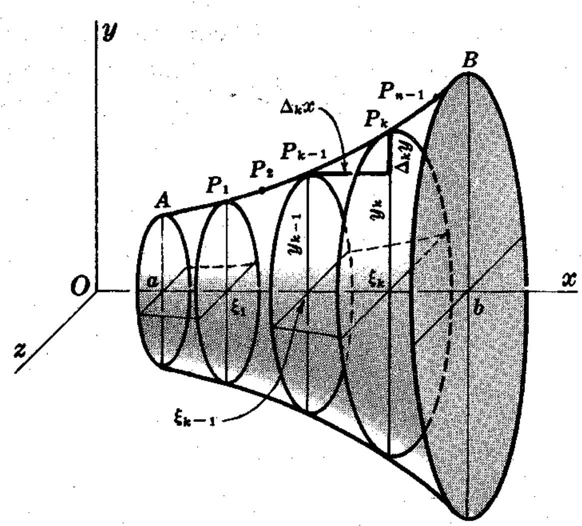

Fig. 48-1

When, in addition, $f'(x) \neq 0$ on the interval, an alternative form of (48.1) is

$$S_x = 2\pi \int_{AB} y\, ds = 2\pi \int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{48.2}$$

If $A(a, c)$ and $B(b, d)$ are two points of the curve $x = g(y)$, where $g(y)$ and its derivative with respect to y satisfy conditions similar to those listed in the previous paragraph, the area of the surface generated by revolving the arc AB about the y axis is given by

$$S_y = 2\pi \int_{AB} x\, ds = 2\pi \int_a^b x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \tag{48.3}$$

If $A(u = u_1)$ and $B(u = u_2)$ are two points on the curve defined by the parametric equations $x = f(u)$, $y = g(u)$ and if conditions of continuity are satisfied, the area of the surface generated by revolving the arc AB about the x axis is given by

$$S_x = 2\pi \int_{AB} y\, ds = 2\pi \int_{u_1}^{u_2} y\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

and the area generated by revolving the arc AB about the y axis is given by

$$S_y = 2\pi \int_{AB} x\, ds = 2\pi \int_{u_1}^{u_2} x\sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

Solved Problems

1. Find the area of the surface of revolution generated by revolving about the x axis the arc of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$. (See Fig. 48-2.)

Solution using (48.1): Here $\dfrac{dy}{dx} = \dfrac{6}{y}$ and $1 + \left(\dfrac{dy}{dx}\right)^2 = \dfrac{y^2 + 36}{y^2}$. Then

$$S_x = 2\pi \int_0^3 y \frac{\sqrt{y^2 + 36}}{y}\, dx = 2\pi \int_0^3 \sqrt{12x + 36}\, dx = 24(2\sqrt{2} - 1)\pi \text{ square units}$$

Solution using (48.2): $\dfrac{dx}{dy} = \dfrac{y}{6}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{36 + y^2}{36}$. Hence,

$$S_x = 2\pi \int_0^6 y \frac{\sqrt{36 + y^2}}{6}\, dy = \left[\frac{\pi}{9}(36 + y^2)^{3/2}\right]_0^6 = 24(2\sqrt{2} - 1)\pi \text{ square units}$$

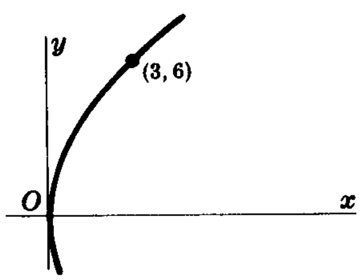

Fig. 48-2

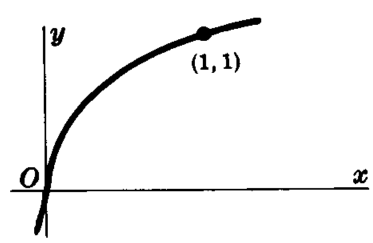

Fig. 48-3

2. Find the area of the surface of revolution generated by revolving about the y axis the arc of $x = y^3$ from $y = 0$ to $y = 1$.

Using (*48.3*) and Fig. 48-3, we have

$$S_y = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_0^1 y^3\sqrt{1 + 9y^4}\, dy = \left[\frac{\pi}{27}(1 + 9y^4)^{3/2}\right]_0^1$$
$$= \frac{\pi}{27}(10\sqrt{10} - 1) \text{ square units}$$

3. Find the area of the surface of revolution generated by revolving about the x axis the arc of $y^2 + 4x = 2\ln y$ from $y = 1$ to $y = 3$.

$$S_x = 2\pi \int_c^d y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_1^3 y\, \frac{1 + y^2}{2y}\, dy = \pi \int_1^3 (1 + y^2)\, dy = \frac{32}{3}\pi \text{ square units}$$

4. Find the area of the surface of revolution generated by revolving a loop of the curve $8a^2y^2 = a^2x^2 - x^4$ about the x axis. (See Fig. 48-4.)

Here $\quad \dfrac{dy}{dx} = \dfrac{a^2x - 2x^3}{8a^2y} \quad$ and $\quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{(a^2 - 2x^2)^2}{8a^2(a^2 - x^2)} = \dfrac{(3a^2 - 2x^2)^2}{8a^2(a^2 - x^2)}$

Hence $\quad S_x = 2\pi \displaystyle\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^a \frac{x\sqrt{a^2 - x^2}}{2a\sqrt{2}}\, \frac{3a^2 - 2x^2}{2a\sqrt{2}\sqrt{a^2 - x^2}}\, dx$

$$= \frac{\pi}{4a^2} \int_0^a (3a^2 - 2x^2)x\, dx = \frac{1}{4}\pi a^2 \text{ square units}$$

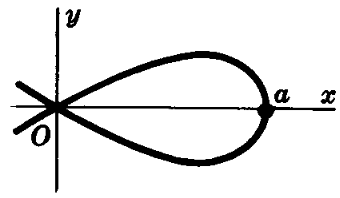

Fig. 48-4

5. Find the area of the surface of revolution generated by revolving about the x axis the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

$$S_x = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_{-4}^{4} y\, \frac{\sqrt{16y^2 + x^2}}{4y}\, dx = \frac{1}{2}\,\pi \int_{-4}^{4} \sqrt{64 - 3x^2}\, dx$$

$$= \frac{\pi}{2\sqrt{3}} \left[\frac{x\sqrt{3}}{2}\sqrt{64 - 3x^2} + 32 \arcsin \frac{x\sqrt{3}}{8}\right]_{-4}^{4} = 8\pi\left(1 + \frac{4\sqrt{3}}{9}\,\pi\right) \text{ square units}$$

6. Find the area of the surface of revolution generated by revolving about the x axis the hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

The required surface is generated by revolving the arc from $\theta = 0$ to $\theta = \pi$. We have $\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$, and $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\cos^2\theta\sin^2\theta$. Then

$$S_x = 2(2\pi)\int_0^{\pi/2} y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 2(2\pi)\int_0^{\pi/2} (a\sin^3\theta) 3a\cos\theta\sin\theta\, d\theta$$

$$= \frac{12a^2\pi}{5} \text{ square units}$$

Note: It would seem natural to write $2\pi\int_0^{\pi} (a\sin^3\theta)3a\cos\theta\sin\theta\, d\theta$ above, but the value then becomes 0. It must be remembered that while areas, volumes, etc., are given by definite integrals, not every definite integral can be interpreted as an area, volume, etc.

7. Find the area of the surface of revolution generated by revolving about the x axis the cardioid $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$.

The required surface is generated by revolving the arc from $\theta = 0$ to $\theta = \pi$ (Fig. 48-5). We have $\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$, $\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$, and so $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 8(1 - \sin\theta\sin 2\theta - \cos_2\theta_2 \cos 2\theta) = 8(1 - \cos\theta)$. Then

$$S_x = 2\pi\int_0^{\pi} (2\sin\theta - \sin 2\theta)(2\sqrt{2}\sqrt{1 - \cos\theta}\, d\theta)$$

$$= 8\sqrt{2}\pi\int_0^{\pi} \sin\theta(1 - \cos\theta)^{3/2}\, d\theta = \left[\frac{16\sqrt{2}}{5}\pi(1 - \cos\theta)^{5/2}\right]_0^{\pi} = \frac{128\pi}{5} \text{ square units}$$

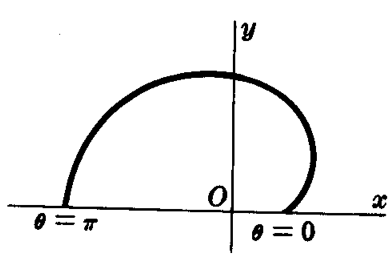

Fig. 48-5

8. Derive: $S_x = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

Let the arc AB be approximated by n chords, as in Fig. 48-1. The representative chord $P_{k-1}P_k$, when revolved about the x axis, generates the frustum of a cone whose bases are of radii y_{k-1} and y_k, whose slant height is

$$P_{k-1}P_k = \sqrt{(\Delta_k x)^2 + (\Delta_k y)^2} = \sqrt{1+\left(\frac{\Delta_k y}{\Delta_k x}\right)^2}\,\Delta_k x = \sqrt{1+[f'(x_k)]^2}\,\Delta_k x$$

(see Problem 1 of Chapter 47), and whose lateral area (circumference of midsection $\times$ slant height) is

$$S_k = 2\pi\,\frac{y_{k-1}+y_k}{2}\sqrt{1+[f'(x_k)]^2}\,\Delta_k x$$

Since $f(x)$ is continuous, there exists at least one point X_k on the arc $P_{k-1}P_k$ such that

$$f(X_k) = \tfrac{1}{2}(y_{k-1}+y_k) = \tfrac{1}{2}[f(\xi_{k-1}) + f(\xi_k)]$$

Hence, $S_k = 2\pi f(X_k)\sqrt{1+[f'(x_k)]^2}\,\Delta_k x$ and, by the theorem of Bliss,

$$S_x = \lim_{n\to+\infty}\sum_{k=1}^{n} S_k = \lim_{n\to+\infty}\sum_{k=1}^{n} 2\pi f(X_k)\sqrt{1+[f'(x_k)]^2}\,\Delta_k x = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Supplementary Problems

In Problems 9 to 18, find the area of the surface generated by revolving the given arc about the given axis. (Answers are in square units.)

9. $y = mx$ from $x=0$ to $x=2$; x axis *Ans.* $4m\pi\sqrt{1+m^2}$

10. $y = \frac{1}{3}x^3$ from $x=0$ to $x=3$; x axis *Ans.* $\pi(82\sqrt{82}-1)/9$

11. $y = \frac{1}{3}x^3$ from $x=0$ to $x=3$; y axis *Ans.* $\frac{1}{2}\pi[9\sqrt{82} + \ln(9+\sqrt{82})]$

12. One loop of $8y^2 = x^2(1-x^2)$; x axis *Ans.* $\frac{1}{4}\pi$

13. $y = x^3/6 + 1/2x$ from $x=1$ to $x=2$; y axis *Ans.* $(\frac{15}{4} + \ln 2)\pi$

14. $y = \ln x$ from $x=1$ to $x=7$; y axis *Ans.* $[34\sqrt{2} + \ln(3+2\sqrt{2})]\pi$

15. One loop of $9y^2 = x(3-x)^2$; y axis *Ans.* $28\pi\sqrt{3}/5$

16. $y = a\cosh x/a$ from $x=-a$ to $x=a$; x axis *Ans.* $\frac{1}{2}\pi a^2(e^2 - e^{-2} + 4)$

17. An arch of $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$; x axis *Ans.* $64\pi a^2/3$

18. $x = e^t\cos t$, $y = e^t\sin t$ from $t=0$ to $t=\frac{1}{2}\pi$; x axis *Ans.* $2\pi\sqrt{2}(2e^{\pi}+1)/5$

19. Find the surface area of a zone cut from a sphere of radius r by two parallel planes, each at a distance $\frac{1}{2}a$ from the center. *Ans.* $2\pi ar$ square units

20. Find the surface area cut from a sphere of radius r by a circular cone of half angle α with its vertex at the center of the sphere. *Ans.* $2\pi r^2(1-\cos\alpha)$ square units

Chapter 49

Centroids and Moments of Inertia of Arcs and Surfaces of Revolution

CENTROID OF AN ARC. The coordinates $(\bar{x}, \bar{y})$ of the centroid of an arc AB of a plane curve of equation $F(x, y) = 0$ or $x = f(u)$, $y = g(u)$ satisfy the relations

$$\bar{x}s = \bar{x}\int_{AB} ds = \int_{AB} x\,ds \qquad \text{and} \qquad \bar{y}s = \bar{y}\int_{AB} ds = \int_{AB} y\,ds$$

(See Problems 1 and 2.)

SECOND THEOREM OF PAPPUS. If an arc of a curve is revolved about an axis in its plane but not crossing the arc, the area of the surface generated is equal to the product of the length of the arc and the length of the path described by the centroid of the arc. (See Problem 3.)

MOMENTS OF INERTIA OF AN ARC. The moments of inertia with respect to the coordinate axes of an arc AB of a curve (a piece of homogeneous fine wire, for example) are given by

$$I_x = \int_{AB} y^2\,ds \qquad \text{and} \qquad I_y = \int_{AB} x^2\,ds$$

(See Problems 4 and 5.)

CENTROID OF A SURFACE OF REVOLUTION. The coordinate $\bar{x}$ of the centroid of the surface generated by revolving an arc AB of a curve about the x axis satisfies the relation

$$\bar{x}S_x = 2\pi \int_{AB} xy\,ds$$

MOMENT OF INERTIA OF A SURFACE OF REVOLUTION. The moment of inertia with respect to the axis of rotation of the surface generated by revolving an arc AB of a curve about the x axis is given by

$$I_x = 2\pi \int_{AB} y^2(y\,ds) = 2\pi \int_{AB} y^3\,ds$$

Solved Problems

1. Find the centroid of the first-quadrant arc of the circle $x^2 + y^2 = 25$. (See Fig. 49-1.)

Here $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{x^2}{y^2} = \dfrac{25}{y^2}$. Since $s = \dfrac{5}{2}\pi$, we have

$$\frac{5}{2}\pi\bar{y} = \int_0^5 y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^5 5\,dx = 25$$

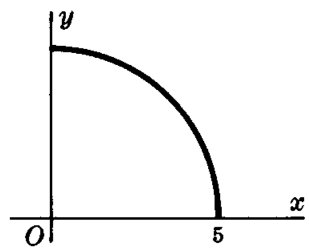

Fig. 49-1

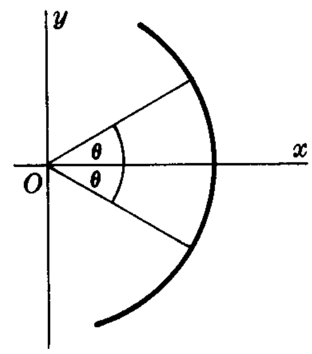

Fig. 49-2

Hence, $\bar{y} = 10/\pi$. By symmetry, $\bar{x} = \bar{y}$ and the coordinates of the centroid are $(10/\pi, 10/\pi)$.

2. Find the centroid of a circular arc of radius r and central angle 2θ.

Take the arc as in Fig. 49-2, so that $\bar{x}$ is identical with the abscissa of the centroid of the upper half of the arc and $\bar{y} = 0$. Then $\dfrac{dx}{dy} = -\dfrac{y}{x}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{r^2}{x^2}$. For the upper half of the arc, $s = r\theta$ and

$$r\theta\bar{x} = \int_0^{r\sin\theta} x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = r\int_0^{r\sin\theta} dy = r^2 \sin\theta$$

Then $\bar{x} = (r\sin\theta)/\theta$. Thus, the centroid is on the bisecting radius at a distance $(r\sin\theta)/\theta$ from the center of the circle.

3. Find the area of the surface generated by revolving the rectangle of dimensions a by b about an axis that is c units from the centroid ($c > a, b$).

The perimeter of the rectangle is $2(a + b)$, and the centroid describes a circle of radius c (Fig. 49-3). Then $S = 2(a + b)(2\pi c) = 4\pi(a + b)c$ square units by the second theorem of Pappus.

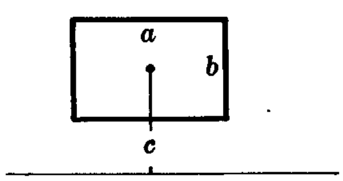

Fig. 49-3

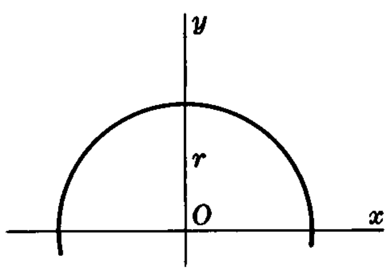

Fig. 49-4

4. Find the moment of inertia of the circumference of a circle with respect to a fixed diameter.

Take the circle as in Fig. 49-4, with the fixed diameter along the x axis. The required moment is four times that of the first-quadrant arc. Since $\dfrac{dy}{dx} = -\dfrac{x}{y}$ and $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{r}{y}$ and $s = 2\pi r$, we have

$$I_x = 4\int_0^r y^2\, ds = 4\int_0^r y^2 \frac{r}{y}\, dx = 4r\int_0^r \sqrt{r^2 - x^2}\, dx$$

$$= 4r\left[\frac{1}{2}x\sqrt{r^2 - x^2} + \frac{1}{2}r^2 \arcsin\frac{x}{r}\right]_0^r = \pi r^3 = \frac{1}{2}r^2 s$$

5. Find the moment of inertia with respect to the x axis of the hypocycloid $x = a\sin^3\theta$, $y = a\cos^3\theta$.

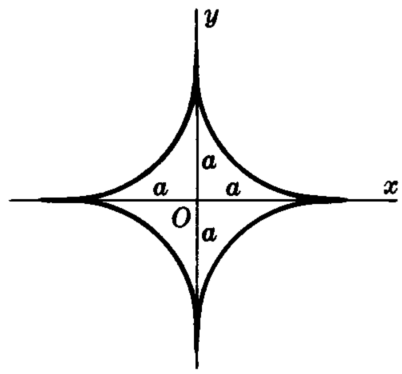

Fig. 49-5

The required moment is four times that of the first-quadrant arc. We have $dx/d\theta = 3a\sin^2\theta\cos\theta$ and $dy/d\theta = -3a\cos^2\theta\sin\theta$, and

$$I_x = 4\int y^2\,ds = 12a^3\int_0^{\pi/2}\cos^6\theta\sin\theta\cos\theta\,d\theta = \tfrac{3}{2}a^3$$

Supplementary Problems

6. Find the centroid of
(a) The first-quadrant arc of $x^{2/3} + y^{2/3} = a^{2/3}$, using $s = 3a/2$ *Ans.* $(2a/5, 2a/5)$
(b) The first-quadrant arc of the loop of $9y^2 = x(3-x)^2$, using $s = 2\sqrt{3}$ *Ans.* $(7/5, \sqrt{3}/4)$
(c) The first arch of $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ *Ans.* $(\pi a, 4a/3)$
(d) The first-quadrant arc of $x = a\cos^3\theta$, $y = a\sin^3\theta$ *Ans.* same as (a)

7. Find the moment of inertia of the given arc with respect to the given line or lines:
(a) Loop of $9y^2 = x(3-x)^2$; x axis, y axis (Use $s = 4\sqrt{3}$.) *Ans.* $I_x = 8s/35$; $I_y = 99s/35$
(b) $y = a\cosh(x/a)$ from $x = 0$ to $x = a$; x axis *Ans.* $(a^2 + \frac{1}{3}s^2)s$

8. Find the centroid of a hemispherical surface. *Ans.* $\bar{y} = \frac{1}{2}r$

9. Find the centroid of the surface generated by revolving
(a) $4y + 3x = 8$ from $x = 0$ to $x = 2$ about the x axis *Ans.* $\bar{x} = 4/5$
(b) An arch of $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about the y axis *Ans.* $\bar{y} = 4a/3$

10. Use the second theorem of Pappus to obtain
(a) The centroid of the first-quadrant arc of a circle of radius r *Ans.* $(2r/\pi, 2r/\pi)$.
(b) The area of the surface generated by revolving an equilateral triangle of side a about an axis that is c units from its centroid. *Ans.* $6\pi ac$ square units

11. Find the moment of inertia with respect to the axis of rotation of
(a) The spherical surface of radius r *Ans.* $\frac{2}{3}Sr^2$
(b) The lateral surface of a cone generated by revolving the line $y = 2x$ from $x = 0$ to $x = 2$ about the x axis *Ans.* $8S$

12. Derive each of the formulas of this chapter.

Chapter 50

Plane Area and Centroid of an Area in Polar Coordinates

THE PLANE AREA bounded by the curve $\rho = f(\theta)$ and the radius vectors $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta$$

When polar coordinates are involved, considerable care must be taken to determine the proper limits of integration. This requires that, by taking advantage of any symmetry, the limits be made as narrow as possible. (See Problems 1 to 7.)

CENTROID OF A PLANE AREA. The coordinates $(\bar{x}, \bar{y})$ of the centroid of a plane area bounded by the curve $\rho = f(\theta)$ and the radius vectors $\theta = \theta_1$ and $\theta = \theta_2$ are given by

$$A\bar{x} = \bar{x}\left(\frac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta\right) = \frac{1}{3}\int_{\theta_1}^{\theta_2} \rho^3 \cos\theta\, d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} \frac{2}{3}x\rho^2\, d\theta$$

and

$$A\bar{y} = \bar{y}\left(\frac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta\right) = \frac{1}{3}\int_{\theta_1}^{\theta_2} \rho^3 \sin\theta\, d\theta = \frac{1}{2}\int_{\theta_1}^{\theta_2} \frac{2}{3}y\rho^2\, d\theta$$

(See Problems 8 to 10.)

Solved Problems

1. Derive $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\, d\theta$.

Let the angle BOC of Fig. 50-1 be divided into n parts by rays $OP_0 = OB, OP_1, OP_2, \ldots, OP_{n-1}, OP_n = OC$. The figure shows a representative slice $P_{k-1}OP_k$ of central angle $\Delta_k\theta$ and its approximating circular sector $R_{k-1}OR_k$ of radius ρ_k, of central angle $\Delta_k\theta$, and (see Problem 15(r) of Chapter 39) of area

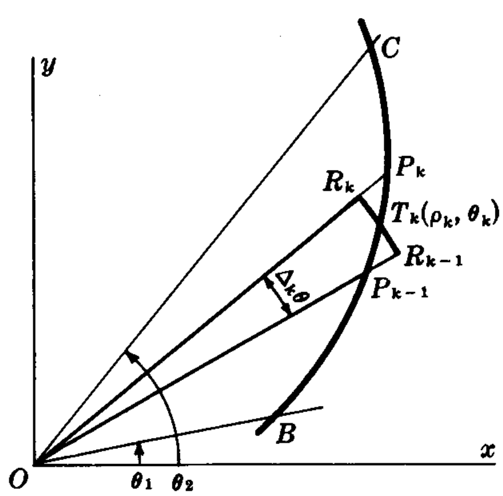

Fig. 50-1

$\frac{1}{2}\rho_k^2\,\Delta_k\theta = \frac{1}{2}[f(\theta_k)]^2\,\Delta_k\theta$. Hence, by the fundamental theorem,

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} \tfrac{1}{2}[f(\theta_k)]^2\,\Delta_k\theta = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} [f(\theta)]^2\,d\theta = \tfrac{1}{2}\int_{\theta_1}^{\theta_2} \rho^2\,d\theta$$

2. Find the area bounded by the curve $\rho^2 = a^2\cos 2\theta$.

From Fig. 50-2 we see that the required area consists of four equal pieces, one of which is swept over as θ varies from $\theta = 0$ to $\theta = \frac{1}{4}\pi$. Thus,

$$A = 4\left(\tfrac{1}{2}\int_0^{\pi/4} \rho^2\,d\theta\right) = 2a^2\int_0^{\pi/4}\cos 2\theta\,d\theta = [a^2\sin 2\theta]_0^{\pi/4} = a^2 \text{ square units}$$

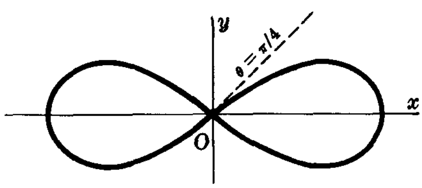

Fig. 50-2

Since portions of the required area lie in each of the quadrants, it might appear reasonable to use, for the required area,

$$\tfrac{1}{2}\int_0^{2\pi} \rho^2\,d\theta = \tfrac{1}{2}a^2\int_0^{2\pi}\cos 2\theta\,d\theta = [\tfrac{1}{4}a^2\sin 2\theta]_0^{2\pi} = 0$$

or

$$2\left(\tfrac{1}{2}\int_0^{\pi} \rho^2\,d\theta\right) = a^2\int_0^{\pi}\cos 2\theta\,d\theta = 0$$

To see why these integrals give incorrect results, consider

$$\tfrac{1}{2}\int_0^{\pi}\rho^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/4}\rho\,d\theta + \tfrac{1}{2}\int_{\pi/4}^{3\pi/4}\rho^2\,d\theta + \tfrac{1}{2}\int_{3\pi/4}^{\pi}\rho^2\,d\theta = \tfrac{1}{4}a^2 - \tfrac{1}{2}a^2 + \tfrac{1}{4}a^2$$

On the intervals $[0, \pi/4]$ and $[3\pi/4, \pi]$, $\rho = a\sqrt{\cos 2\theta}$ is real; thus the first and third integrals give the areas swept over as θ ranges over these intervals. But on the interval $[\pi/4, 3\pi/4]$, $\rho^2 < 0$ and ρ is imaginary. Thus, while $\frac{1}{2}\int_{\pi/4}^{3\pi/4} a^2\cos 2\theta\,d\theta$ is a perfectly valid integral, it cannot be interpreted here as an area.

3. Find the area bounded by the three-leaved rose $\rho = a\cos 3\theta$.

The required area is six times the shaded area in Fig. 50-3, that is, the area swept over as θ varies from 0 to $\pi/6$. Hence,

$$A = 6\left(\tfrac{1}{2}\int_{\theta_1}^{\theta_2}\rho^2\,d\theta\right) = 3\int_0^{\pi/6} a^2\cos^2 3\theta\,d\theta = 3a^2\int_0^{\pi/6}(\tfrac{1}{2} + \tfrac{1}{2}\cos 6\theta)\,d\theta = \tfrac{1}{4}\pi a^2 \text{ square units}$$

4. Find the area bounded by the limacon $\rho = 2 + \cos\theta$ in Fig. 50-4.

The required area is twice that swept over as θ varies from 0 to π:

$$A = 2\left[\frac{1}{2}\int_0^{\pi}(2+\cos\theta)^2\,d\theta\right] = \int_0^{\pi}(4 + 4\cos\theta + \cos^2\theta)\,d\theta$$

$$= \left[4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi} = \frac{9\pi}{2} \text{ square units}$$

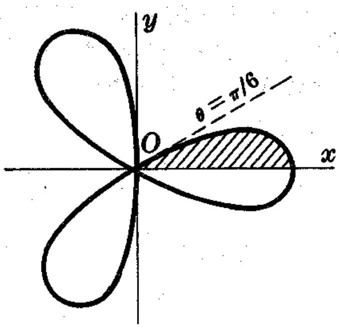

Fig. 50-3

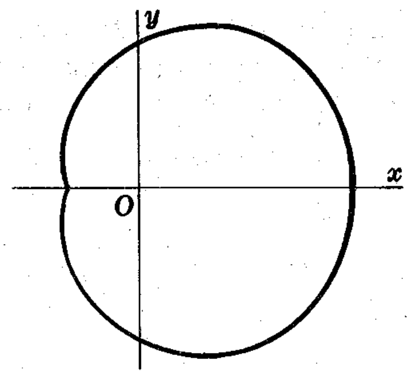

Fig. 50-4

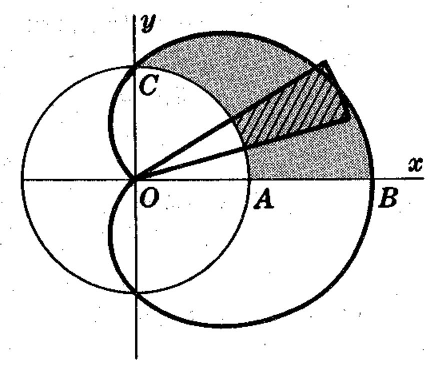

Fig. 50-5

5. Find the area inside the cardioid $\rho = 1 + \cos\theta$ and outside the circle $\rho = 1$.

In Fig. 50-5, area ABC = area OBC − area OAC is one-half the required area. Thus,

$$A = 2\left[\tfrac{1}{2}\int_0^{\pi/2}(1+\cos\theta)^2\,d\theta\right] - 2\left[\tfrac{1}{2}\int_0^{\pi/2}(1)^2\,d\theta\right]$$
$$= \int_0^{\pi/2}(2\cos\theta + \cos^2\theta)\,d\theta = 2 + \tfrac{1}{4}\pi \text{ square units}$$

6. Find the area of each loop of $\rho = \frac{1}{2} + \cos\theta$. (See Fig. 50-6.)

Larger loop: The required area is twice that swept over as θ varies from 0 to $2\pi/3$. Hence,

$$A = 2\left[\frac{1}{2}\int_0^{2\pi/3}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta\right] = \int_0^{2\pi/3}\left(\frac{1}{4}+\cos\theta+\cos^2\theta\right)d\theta = \frac{\pi}{2}+\frac{3\sqrt{3}}{8} \text{ square units}$$

Smaller loop: The required area is twice that swept over as θ varies from $2\pi/3$ to π. Hence,

$$A = 2\left[\frac{1}{2}\int_{2\pi/3}^{\pi}\left(\frac{1}{2}+\cos\theta\right)^2 d\theta\right] = \frac{\pi}{4}-\frac{3\sqrt{3}}{8} \text{ square units}$$

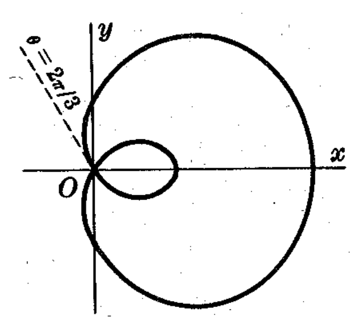

Fig. 50-6

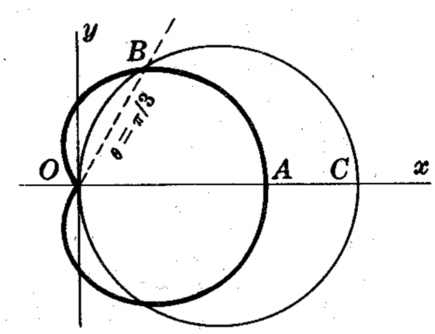

Fig. 50-7

7. Find the area common to the circle $\rho = 3\cos\theta$ and the cardioid $\rho = 1 + \cos\theta$.

Area OAB in Fig. 50-7 consists of two portions, one swept over by the radius vector $\rho = 1 + \cos\theta$ as θ varies from 0 to $\pi/3$, and the other swept over by $\rho = 3\cos\theta$ as θ varies from $\pi/3$ to $\pi/2$. Hence

$$A = 2\left[\frac{1}{2}\int_0^{\pi/3}(1+\cos\theta)^2\,d\theta\right] + 2\left[\frac{1}{2}\int_{\pi/3}^{\pi/2} 9\cos^2\theta\,d\theta\right] = \frac{5\pi}{4} \text{ square units}$$

8. Derive the formulas $A\bar{x} = \frac{1}{3}\int_{\theta_1}^{\theta_2}\rho^3\cos\theta\,d\theta$, $A\bar{y} = \frac{1}{3}\int_{\theta_1}^{\theta_2}\rho^3\sin\theta\,d\theta$, where $(\bar{x}, \bar{y})$ are the coordinates of the centroid of the plane area BOC of Fig. 50-1.

Consider the representative approximating circular sector $R_{k-1}OR_k$ and suppose, for convenience, that OT_k bisects the angle $P_{k-1}OP_k$. To approximate the centroid $C_k(\bar{x}_k, \bar{y}_k)$ of this sector, consider it to be a true triangle. Then its centroid will lie on OT_k at a distance $\frac{2}{3}\rho_k$ from O; thus, approximately,

$$\bar{x}_k = \tfrac{2}{3}\rho_k \cos\theta_k = \tfrac{2}{3}f(\theta_k)\cos\theta_k \qquad \text{and} \qquad \bar{y}_k = \tfrac{2}{3}f(\theta_k)\sin\theta_k$$

Now the first moment of the sector about the y axis is

$$\bar{x}_k(\tfrac{1}{2}\rho_k^2\,\Delta_k\theta) = \tfrac{2}{3}\rho_k\cos\theta_k(\tfrac{1}{2}\rho_k^2\,\Delta_k\theta) = \tfrac{1}{3}[f(\theta_k)]^3\cos\theta_k\,\Delta_k\theta$$

and, by the fundamental theorem,

$$A\bar{x} = \lim_{n\to+\infty}\sum_{k=1}^{n}\tfrac{1}{3}[f(\theta_k)]^3\cos\theta_k\,\Delta_k\theta = \tfrac{1}{3}\int_{\theta_1}^{\theta_2}\rho^3\cos\theta\,d\theta$$

It is left as an exercise to obtain the formula for $A\bar{y}$.

Note: From Problem 8 of Chapter 42, the centroid of the sector $R_{k-1}OR_k$ lies on OT_k at a distance $\dfrac{2\rho_k\sin\frac{1}{2}\Delta_k\theta}{3(\frac{1}{2}\Delta_k\theta)}$ from O. You may wish to use this to derive the formulas.

9. Find the centroid of the area of the first-quadrant loop of the rose $\rho = \sin 2\theta$, shown in Fig. 50-8.

$$A = \frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \frac{1}{4}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{8}$$

So
$$\frac{\pi}{8}\bar{x} = \frac{1}{3}\int_0^{\pi/2}\rho^3\cos\theta\,d\theta = \frac{1}{3}\int_0^{\pi/2}\sin^3 2\theta\cos\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}\sin^3\theta\cos^4\theta\,d\theta$$
$$= \frac{8}{3}\int_0^{\pi/2}(1-\cos^2\theta)\cos^4\theta\sin\theta\,d\theta = \frac{16}{105}$$

from which $\bar{x} = 128/105\pi$. By symmetry, $\bar{y} = 128/105\pi$. The coordinates of the centroid are $(128/105\pi, 128/105\pi)$.

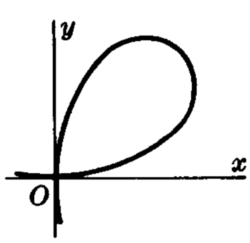

Fig. 50-8

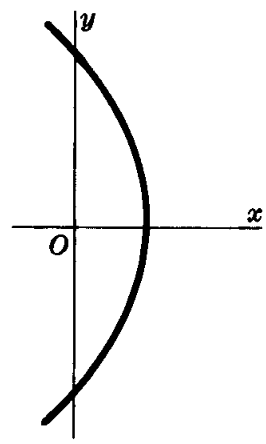

Fig. 50-9

10. Find the centroid of the first-quadrant area bounded by the parabola $\rho = \dfrac{6}{1+\cos\theta}$ in Fig. 50-9.

$$A = \frac{1}{2}\int_0^{\pi/2}\frac{36}{(1+\cos\theta)^2}\,d\theta = \frac{9}{2}\int_0^{\pi/2}\sec^4\frac{1}{2}\theta\,d\theta$$
$$= \frac{9}{2}\int_0^{\pi/2}\left(1+\tan^2\frac{1}{2}\theta\right)\sec^2\frac{1}{2}\theta\,d\theta = 9\left[\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta\right]_0^{\pi/2} = 12$$

So
$$12\bar{x} = \frac{1}{3}\int_0^{\pi/2}\frac{216\cos\theta}{(1+\cos\theta)^3}\,d\theta = 9\int_0^{\pi/2}\frac{2\cos^2(\theta/2)-1}{\cos^6\theta/2}\,d\theta = 9\int_0^{\pi/2}\left(2\sec^4\frac{\theta}{2} - \sec^6\frac{\theta}{2}\right)d\theta$$

$$= 18\left[\tan\frac{\theta}{2} - \frac{1}{5}\tan^5\frac{\theta}{2}\right]_0^{\pi/2} = \frac{72}{5}$$

and $$12\bar{y} = \frac{1}{3}\int_0^{\pi/2} \frac{216\sin\theta}{(1+\cos\theta)^3}\,d\theta = 27$$

Hence $\bar{x} = \frac{6}{5}$ and $\bar{y} = \frac{9}{4}$, and the centroid is $(6/5, 9/4)$.

Supplementary Problems

11. Find the area bounded by each of the following curves. (Answers are in square units.)
(*a*) $\rho^2 = 1 + \cos 2\theta$ *Ans.* π (*b*) $\rho^2 = a^2 \sin\theta(1-\cos\theta)$ *Ans.* a^2
(*c*) $\rho = 4\cos\theta$ *Ans.* 4π (*d*) $\rho = a\cos 2\theta$ *Ans.* $\frac{1}{2}\pi a^2$
(*e*) $\rho = 4\sin^2\theta$ *Ans.* 6π (*f*) $\rho = 4(1-\sin\theta)$ *Ans.* 24π

12. Find the area described in each of the following. (Answers are in square units.)
(*a*) Inside $\rho = \cos\theta$ and outside $\rho = 1 - \cos\theta$ *Ans.* $(\sqrt{3} - \pi/3)$
(*b*) Inside $\rho = \sin\theta$ and outside $\rho = 1 - \cos\theta$ *Ans.* $(1 - \pi/4)$
(*c*) Between the inner and outer ovals of $\rho^2 = a^2(1 + \sin\theta)$ *Ans.* $4a^2$
(*d*) Between the loops of $\rho = 2 - 4\sin\theta$ *Ans.* $4(\pi + 3\sqrt{3})$

13. (*a*) For the spiral of Archimedes, $\rho = a\theta$, show that the additional area swept over by the nth revolution, for $n > 2$, is $n - 1$ times that added by the second revolution.
(*b*) For the equiangular spiral $\rho = ae^{\theta}$, show that the additional area swept over by the nth revolution, for $n > 2$, is $e^{4\pi}$ times that added by the previous revolution.

14. Find the centroids of the following areas:
(*a*) Right half of $\rho = a(1 - \sin\theta)$ *Ans.* $(16a/9\pi, -5a/6)$
(*b*) First-quadrant area of $\rho = 4\sin^2\theta$ *Ans.* $(128/63\pi, 2048/315\pi)$
(*c*) Upper half of $\rho = 2 + \cos\theta$ *Ans.* $(17/18, 80/27\pi)$
(*d*) First-quadrant area of $\rho = 1 + \cos\theta$ *Ans.* $\left(\dfrac{16+5\pi}{16+6\pi}, \dfrac{10}{8+3\pi}\right)$
(*e*) First-quadrant area of Problem 5. *Ans.* $\left(\dfrac{32+15\pi}{48+6\pi}, \dfrac{22}{24+3\pi}\right)$

15. Use the first theorem of Pappus to obtain the volume generated by revolving
(*a*) $\rho = a(1 - \sin\theta)$ about the 90° line *Ans.* $8\pi a^3/3$ cubic units
(*b*) $\rho = 2 + \cos\theta$ about the polar axis *Ans.* $40\pi/3$ cubic units

Chapter 51

Length and Centroid of an Arc and Area of a Surface of Revolution in Polar Coordinates

THE LENGTH OF THE ARC of the curve $\rho = f(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$ is given by

$$s = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta$$

(See Problems 1 to 4.)

CENTROID OF AN ARC. The coordinates $(\bar{x}, \bar{y})$ of the centroid of the arc of the curve $\rho = f(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$ satisfy the relations

$$\bar{x}s = \bar{x}\int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \rho\cos\theta\, ds = \int_{\theta_1}^{\theta_2} x\, ds$$

$$\bar{y}s = \bar{y}\int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \rho\sin\theta\, ds = \int_{\theta_1}^{\theta_2} y\, ds$$

(See Problems 5 and 6.)

THE AREA OF THE SURFACE generated by revolving the arc of the curve $\rho = f(\theta)$ from $\theta = \theta_1$ to $\theta = \theta_2$ about the polar axis is

$$S_x = 2\pi\int_{\theta_1}^{\theta_2} y\, ds = 2\pi\int_{\theta_1}^{\theta_2} \rho\sin\theta\, ds$$

and about the 90° line is

$$S_y = 2\pi\int_{\theta_1}^{\theta_2} x\, ds = 2\pi\int_{\theta_1}^{\theta_2} \rho\cos\theta\, ds$$

The limits of integration should be taken as narrowly as possible. (See Problems 7 to 10.)

Solved Problems

1. Find the length of the spiral $\rho = e^{2\theta}$ from $\theta = 0$ to $\theta = 2\pi$ (Fig. 51-1).

Here $d\rho/d\theta = 2e^{2\theta}$ and $\rho^2 + (d\rho/d\theta)^2 = 5e^{4\theta}$. Hence

$$s = \int_0^{2\pi} \sqrt{\rho^2 + (d\rho/d\theta)^2}\, d\theta = \sqrt{5}\int_0^{2\pi} e^{2\theta}\, d\theta = \tfrac{1}{2}\sqrt{5}(e^{4\pi} - 1) \text{ units}$$

2. Find the length of the cardioid $\rho = a(1 - \cos\theta)$.

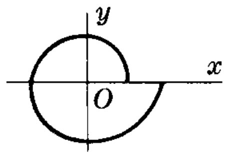

Fig. 51-1

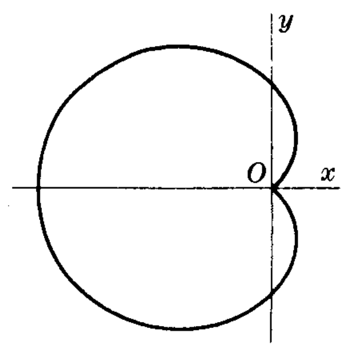

Fig. 51-2

The cardioid is described as θ varies from 0 to 2π (see Fig. 51-2). Since $\rho^2 + (d\rho/d\theta)^2 = a^2(1 - \cos\theta)^2 + (a \sin\theta)^2 = 4a^2 \sin^2 \frac{1}{2}\theta$, we have

$$s = \int_0^{2\pi} \sqrt{\rho^2 + (d\rho/d\theta)^2}\, d\theta = 2a \int_0^{2\pi} \sin \tfrac{1}{2}\theta\, d\theta = 8a \text{ units}$$

In this solution the instruction to take the limits of integration as narrowly as possible has not been followed, since the required length could be obtained as twice that described as θ varies from 0 to π. However, see Problem 3 below.

3. Find the length of the cardioid $\rho = a(1 - \sin\theta)$, shown in Fig. 51-3.

Here $\rho^2 + (d\rho/d\theta)^2 = a^2(1 - \sin\theta)^2 + (-a\cos\theta)^2 = 2a^2(\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta)^2$. Following Problem 2, we write

$$s = \int_0^{2\pi} \sqrt{\rho^2 + (d\rho/d\theta)^2}\, d\theta = \sqrt{2}a \int_0^{2\pi} (\sin\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta)\, d\theta$$
$$= [2\sqrt{2}a(-\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)]_0^{2\pi} = 4\sqrt{2}a \text{ units}$$

The cardioids of the two problems differ only in their positions in the plane; hence their lengths should agree. An explanation for the disagreement is to be found in a comparison of the two integrands $\sin\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta - \cos\frac{1}{2}\theta$. The first is never negative, while the second is negative as θ varies from 0 to $\frac{1}{2}\pi$ and positive otherwise. By symmetry, the required length in this problem is twice that described as θ varies from $\pi/2$ to $3\pi/2$. It may be found as

$$s = 2\sqrt{2}a \int_{\pi/2}^{3\pi/2} (\sin\tfrac{1}{2}\theta - \cos\tfrac{1}{2}\theta)\, d\theta = [4\sqrt{2}a(-\cos\tfrac{1}{2}\theta - \sin\tfrac{1}{2}\theta)]_{\pi/2}^{3\pi/2} = 8a \text{ units}$$

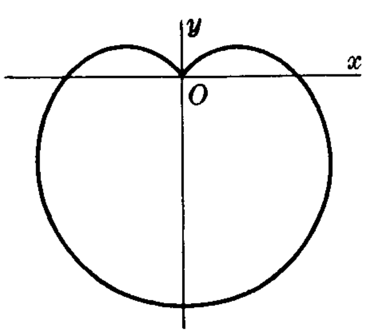

Fig. 51-3

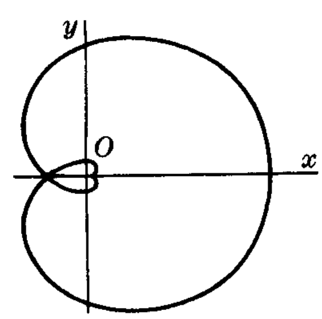

Fig. 51-4

4. Find the length of the curve $\rho = a\cos^4 \frac{1}{4}\theta$.

The required length is twice that described as θ varies from 0 to 2π in Fig. 51-4. We have $d\rho/d\theta = -a\cos^3\frac{1}{4}\theta \sin\frac{1}{4}\theta$ and $\rho^2 + (d\rho/d\theta)^2 = a^2\cos^6\frac{1}{4}\theta$. Hence,

$$s = 2\left(a \int_0^{2\pi} \cos^3 \tfrac{1}{4}\theta\, d\theta\right) = 8a[\sin\tfrac{1}{4}\theta - \tfrac{1}{3}\sin^3\tfrac{1}{4}\theta]_0^{2\pi} = \tfrac{16}{3}a \text{ units}$$

5. Find the centroid of the arc of the cardioid $\rho = a(1 - \cos\theta)$. Refer to Problem 2 and Fig. 51-2.

By symmetry, $\bar{y} = 0$ and the abscissa of the centroid of the entire arc is the same as that for the upper half. From Problem 2, half the length of the cardioid is $4a$; hence,

$$4a\bar{x} = \int_0^\pi \rho\cos\theta\sqrt{\rho^2 + (d\rho/d\theta)^2}\,d\theta = 2a^2\int_0^\pi (1 - \cos\theta)\cos\theta\sin\tfrac{1}{2}\theta\,d\theta$$

$$= 4a^2\int_0^\pi (-2\cos^4\tfrac{1}{2}\theta + 3\cos^2\tfrac{1}{2}\theta - 1)\sin\tfrac{1}{2}\theta\,d\theta = 4a^2[\tfrac{4}{5}\cos^5\tfrac{1}{2}\theta - 2\cos^3\tfrac{1}{2}\theta + 2\cos\tfrac{1}{2}\theta]_0^\pi = \tfrac{16}{5}a^2$$

and $\bar{x} = -4a/5$. The coordinates of the centroid are $(-4a/5, 0)$.

6. Find the centroid of the arc of the circle $\rho = 2\sin\theta + 4\cos\theta$ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$.

We can see that the curve is a circle passing through the origin with center (2, 1) and radius $\sqrt{5}$ (see Fig. 51-5) by noting that $x^2 + y^2 = \rho^2 = 2\rho\sin\theta + 4\rho\cos\theta = 2y + 4x$, which simplifies to $(x-2)^2 + (y-1)^2 = 5$. Also, $d\rho/d\theta = 2\cos\theta - 4\sin\theta$ and $\rho^2 + (d\rho/d\theta)^2 = 20$. Since the radius is $\sqrt{5}$, $s = \sqrt{5}\pi$. Then

$$\sqrt{5}\pi\bar{x} = \int_0^{\pi/2} \rho\cos\theta\sqrt{\rho^2 + (d\rho/d\theta)^2}\,d\theta = 4\sqrt{5}\int_0^{\pi/2} (\sin\theta\cos\theta + 2\cos^2\theta)\,d\theta$$

$$= 4\sqrt{5}[\tfrac{1}{2}\sin^2\theta + \theta + \tfrac{1}{2}\sin 2\theta]_0^{\pi/2} = 2\sqrt{5}(\pi + 1)$$

and

$$\sqrt{5}\pi\bar{y} = \int_0^{\pi/2} \rho\sin\theta\sqrt{\rho^2 + (d\rho/d\theta)^2}\,d\theta = 4\sqrt{5}\int_0^{\pi/2} (\sin^2\theta + 2\sin\theta\cos\theta)\,d\theta$$

$$= 4\sqrt{5}[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + \sin^2\theta]_0^{\pi/2} = 4\sqrt{5}(\tfrac{1}{4}\pi + 1)$$

Hence $\bar{x} = 2(\pi + 1)/\pi$ and $\bar{y} = (\pi + 4)/\pi$.

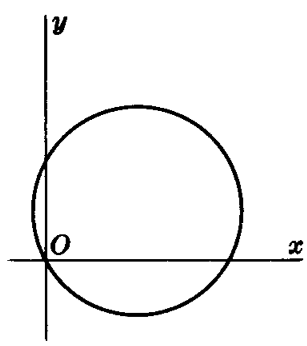

Fig. 51-5

7. Find the area of the surface generated by revolving the upper half of the cardioid $\rho = a(1 - \cos\theta)$ about the polar axis.

From Problem 2, $\rho^2 + (d\rho/d\theta)^2 = 4a^2\sin^2\frac{1}{2}\theta$. Then

$$S_x = 2\pi\int_0^\pi \rho\sin\theta\sqrt{\rho^2 + (d\rho/d\theta)^2}\,d\theta = 4a^2\pi\int_0^\pi (1 - \cos\theta)\sin\theta\sin\tfrac{1}{2}\theta\,d\theta$$

$$= 16a^2\pi\int_0^\pi \sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta\,d\theta = \tfrac{32}{5}a^2\pi \text{ square units}$$

8. Find the area of the surface generated by revolving the lemniscate $\rho^2 = a^2\cos 2\theta$ about the polar axis.

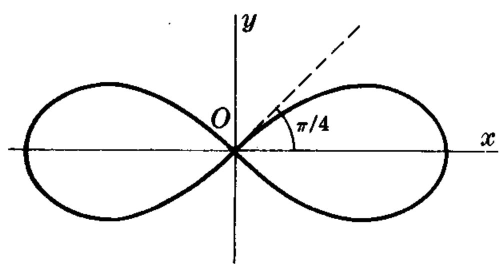

Fig. 51-6

The required area is twice that generated by revolving the first-quadrant arc (see Fig. 51-6). Since

$$\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2 = a^2 \cos 2\theta + \left(-\frac{a^2 \sin 2\theta}{\rho}\right)^2 = \frac{a^4}{\rho^2}$$

$$S_x = 2\left(2\pi \int_0^{\pi/4} \rho \sin\theta \frac{a^2}{\rho}\, d\theta\right) = 4a^2\pi \int_0^{\pi/4} \sin\theta\, d\theta = 2a^2\pi(2-\sqrt{2}) \text{ square units}$$

9. Find the area of the surface generated by revolving a loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ about the 90° line.

The required area is twice that generated by revolving the first-quadrant arc:

$$S_y = 2\left(2\pi \int_0^{\pi/4} \rho \cos\theta \frac{a^2}{\rho}\, d\theta\right) = 4a^2\pi \int_0^{\pi/4} \cos\theta\, d\theta = 2\sqrt{2}a^2\pi \text{ square units}$$

10. Use the second theorem of Pappus to find the centroid of the arc of the cardioid $\rho = a(1-\cos\theta)$ from $\theta = 0$ to $\theta = \pi$.

If the arc is revolved about the polar axis, then according to the theorem, $S = 2\pi\bar{y}s$. Substituting from Problems 2 and 7 yields $32a^2\pi/5 = 2\pi\bar{y}(4a)$, from which $\bar{y} = 4a/5$. By Problem 5, $\bar{x} = -4a/5$ and so the centroid has coordinates $(-4a/5, 4a/5)$.

Supplementary Problems

11. Find the length of each of the following arcs.

(*a*) $\rho = \theta^2$ from $\theta = 0$ to $\theta = 2\sqrt{3}$ *Ans.* $56/3$ units

(*b*) $\rho = e^{\theta/2}$ from $\theta = 0$ to $\theta = 8$ *Ans.* $\sqrt{5}(e^4 - 1)$ units

(*c*) $\rho = \cos^2(\theta/2)$ *Ans.* 4 units

(*d*) $\rho = \sin^3(\theta/3)$ *Ans.* $3\pi/2$ units

(*e*) $\rho = \cos^4(\theta/4)$ *Ans.* $16/3$ units

(*f*) $\rho = a/\theta$ from (ρ_1, θ_1) to (ρ_2, θ_2) *Ans.* $\sqrt{a^2+\rho_1^2} - \sqrt{a^2+\rho_2^2} + a \ln \dfrac{\rho_1(a+\sqrt{a^2+\rho_2^2})}{\rho_2(a+\sqrt{a^2+\rho_1^2})}$ units

(*g*) $\rho = 2a \tan\theta \sin\theta$ from $\theta = 0$ to $\theta = \pi/3$ *Ans.* $2a\sqrt{3}\left[\dfrac{\sqrt{7}-2}{\sqrt{3}} + \ln \dfrac{2(2+\sqrt{3})}{\sqrt{7}+\sqrt{3}}\right]$ units

12. Find the centroid of the upper half of $\rho = 8\cos\theta$. *Ans.* $(4, 8/\pi)$

13. For $\rho = a\sin\theta + b\cos\theta$, show that $s = \pi\sqrt{a^2+b^2}$, $S_x = a\pi s$, and $S_y = b\pi s$.

14. Find the area of the surface generated by revolving $\rho = 4\cos\theta$ about the polar axis.

Ans. 16π square units

15. Find the area of the surface generated by revolving each loop of $\rho = \sin^3(\theta/3)$ about the 90° line.

Ans. $\pi/256$ square units; $513\pi/256$ square units

16. Find the area of the surface generated by revolving one loop of $\rho^2 = \cos 2\theta$ about the 90° line.

Ans. $2\sqrt{2}\pi$ square units

17. Show that when the two loops of $\rho = \cos^4(\theta/4)$ are revolved about the polar axis, they generate equal surface areas.

18. Find the centroid of the surface generated by revolving the right-hand loop of $\rho^2 = a^2\cos 2\theta$ about the polar axis. *Ans.* $\bar{x} = \sqrt{2}a(\sqrt{2}+1)/6$

19. Find the area of the surface generated by revolving $\rho = \sin^2(\theta/2)$ about the line $\rho = \csc\theta$.

Ans. 8π square units

20. Derive the formulas of this chapter.

Chapter 52

Improper Integrals

THE DEFINITE INTEGRAL $\int_a^b f(x)\,dx$ is called an *improper integral* if either

1. The integrand $f(x)$ has one or more points of discontinuity on the interval $a \leqq x \leqq b$, or
2. At least one of the limits of integration is infinite.

DISCONTINUOUS INTEGRAND. If $f(x)$ is continuous on the interval $a \leqq x < b$ but is discontinuous at $x = b$, we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_a^{b-\epsilon} f(x)\,dx \quad \text{provided the limit exists}$$

If $f(x)$ is continuous on the interval $a < x \leqq b$ but is discontinuous at $x = a$, we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b} f(x)\,dx \quad \text{provided the limit exists}$$

If $f(x)$ is continuous for all values of x on the interval $a \leqq x \leqq b$ except at $x = c$, where $a < c < b$, we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0^+} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon'\to 0^+} \int_{c+\epsilon'}^{b} f(x)\,dx \quad \text{provided } \textit{both} \text{ limits exist}$$

(See Problems 1 to 6.)

INFINITE LIMITS OF INTEGRATION. If $f(x)$ is continuous on every interval $a \leqq x \leqq U$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{U\to +\infty} \int_a^{U} f(x)\,dx \quad \text{provided the limit exists}$$

If $f(x)$ is continuous on every interval $u \leqq x \leqq b$, we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{u\to -\infty} \int_u^{b} f(x)\,dx \quad \text{provided the limit exists}$$

If $f(x)$ is continuous, we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{U\to +\infty} \int_a^{U} f(x)\,dx + \lim_{u\to -\infty} \int_u^{a} f(x)\,dx \quad \text{provided } \textit{both} \text{ limits exist}$$

(See Problems 7 to 13.)

Solved Problems

1. Evaluate $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$. The integrand is discontinuous at $x = 3$. We consider

$$\lim_{\epsilon\to 0^+} \int_0^{3-\epsilon} \frac{dx}{\sqrt{9-x^2}} = \lim_{\epsilon\to 0^+} \left[\arcsin \frac{x}{3}\right]_0^{3-\epsilon} = \lim_{\epsilon\to 0^+} \arcsin \frac{3-\epsilon}{3} = \arcsin 1 = \frac{1}{2}\pi$$

Hence, $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \frac{1}{2}\pi$.

2. Show that $\displaystyle\int_0^2 \frac{dx}{2-x}$ is meaningless.

The integrand is discontinuous at $x = 2$. We consider

$$\lim_{\epsilon\to 0^+}\int_0^{2-\epsilon}\frac{dx}{2-x} = \lim_{\epsilon\to 0^+}\left[\ln\frac{1}{2-x}\right]_0^{2-\epsilon} = \lim_{\epsilon\to 0^+}\left(\ln\frac{1}{\epsilon} - \ln\frac{1}{2}\right)$$

The limit does not exist; so the integral is meaningless.

3. Show that $\displaystyle\int_0^4 \frac{dx}{(x-1)^2}$ is meaningless.

The integrand is discontinuous at $x = 1$, a value between the limits of integration 0 and 4 (see Fig. 52-1). We consider

$$\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{dx}{(x-1)^2} + \lim_{\epsilon'\to 0^+}\int_{1+\epsilon'}^4\frac{dx}{(x-1)^2} = \lim_{\epsilon\to 0^+}\left[\frac{-1}{x-1}\right]_0^{1-\epsilon} + \lim_{\epsilon'\to 0^+}\left[\frac{-1}{x-1}\right]_{1+\epsilon'}^4$$

$$= \lim_{\epsilon\to 0^+}\left(\frac{1}{\epsilon} - 1\right) + \lim_{\epsilon'\to 0^+}\left(-\frac{1}{3} + \frac{1}{\epsilon'}\right)$$

These limits do not exist.

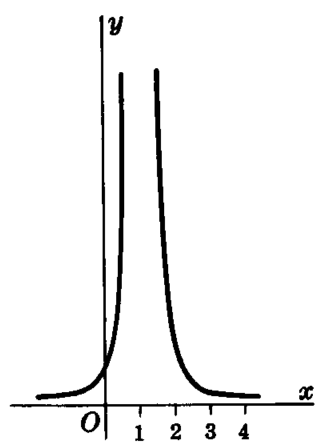

Fig. 52-1

If the point of discontinuity is overlooked, we obtain $\displaystyle\int_0^4 \frac{dx}{(x-1)^2} = \left[-\frac{1}{x-1}\right]_0^4 = -\frac{4}{3}$. This result is absurd because $1/(x-1)^2$ is always positive.

4. Evaluate $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{x-1}}$.

The integrand is discontinuous at $x = 1$. We consider

$$\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{dx}{\sqrt[3]{x+1}} + \lim_{\epsilon'\to 0^+}\int_{1+\epsilon'}^4\frac{dx}{\sqrt[3]{x-1}} = \lim_{\epsilon\to 0^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^{1-\epsilon} + \lim_{\epsilon'\to 0^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_{1+\epsilon'}^4$$

$$= \lim_{\epsilon\to 0^+}\frac{3}{2}\left((-\epsilon)^{2/3} - 1\right) + \lim_{\epsilon'\to 0^+}\frac{3}{2}\left(\sqrt[3]{9} - \epsilon'^{2/3}\right) = \frac{3}{2}\left(\sqrt[3]{9} - 1\right)$$

Hence, $\displaystyle\int_0^4 \frac{dx}{\sqrt[3]{x-1}} = \frac{3}{2}(\sqrt[3]{9} - 1)$.

5. Show that $\int_0^{\pi/2} \sec x\, dx$ is meaningless.

The integrand is discontinuous at $x = \frac{1}{2}\pi$. We consider

$$\lim_{\epsilon\to 0^+} \int_0^{\frac{\pi}{2}-\epsilon} \sec x\, dx = \lim_{\epsilon\to 0^+} [\ln(\sec x + \tan x)]_0^{\frac{\pi}{2}-\epsilon} = \lim_{\epsilon\to 0^+} \ln[\sec(\tfrac{1}{2}\pi - \epsilon) + \tan(\tfrac{1}{2}\pi - \epsilon)]$$

The limit does not exist, so the integral is meaningless.

6. Evaluate $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx$.

The integrand is discontinuous at $x = \frac{1}{2}\pi$. We consider

$$\lim_{\epsilon\to 0^+} \int_0^{\pi-\epsilon} \frac{\cos x}{\sqrt{1-\sin x}}\, dx = \lim_{\epsilon\to 0^+} [-2(1-\sin x)^{\pi}]_0^{\frac{\pi}{2}-\epsilon} = 2 \lim_{\epsilon\to 0^+} \{-[1-\sin(\tfrac{1}{2}\pi - \epsilon)] + 1\} = 2$$

Hence, $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx = 2$.

7. Evaluate $\int_0^{+\infty} \frac{dx}{x^2+4}$.

The upper limit of integration is infinite. We consider

$$\lim_{U\to+\infty} \int_0^U \frac{dx}{x^2+4} = \lim_{U\to+\infty} \left[\frac{1}{2}\arctan\frac{1}{2}x\right]_0^U = \frac{\pi}{4} \qquad \text{from which} \qquad \int_0^{+\infty} \frac{dx}{x^2+4} = \frac{\pi}{4}$$

8. Evaluate $\int_{-\infty}^0 e^{2x}\, dx$.

The lower limit of integration is infinite. We consider

$$\lim_{u\to-\infty} \int_u^0 e^{2x}\, dx = \lim_{u\to-\infty} \left[\frac{1}{2}e^{2x}\right]_u^0 = \frac{1}{2}(1) - \lim_{u\to-\infty} \frac{1}{2}e^{2u} = \frac{1}{2} - 0$$

Hence, $\int_{-\infty}^0 e^{2x}\, dx = \frac{1}{2}$.

9. Show that $\int_1^{+\infty} dx\sqrt{x}$ is meaningless.

The upper limit of integration is infinite. We consider $\lim_{U\to+\infty} \int_1^U dx/\sqrt{x} = \lim_{U\to+\infty} [2\sqrt{x}]_1^U = \lim_{U\to+\infty} (2\sqrt{u} - 2)$. The limit does not exist.

10. Evaluate $\int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{-x}} = \int_{-\infty}^{+\infty} \frac{e^x\, dx}{e^{2x}+1}$.

Both limits of integration are infinite. We consider

$$\begin{aligned}
\lim_{U\to+\infty} \int_0^U \frac{e^x\, dx}{e^{2x}+1} + \lim_{u\to-\infty} \int_u^0 \frac{e^x\, dx}{e^{2x}+1} &= \lim_{U\to+\infty} [\arctan e^x]_0^U + \lim_{u\to-\infty} [\arctan e^x]_u^0 \\
&= \lim_{U\to+\infty} (\arctan e^U - \tfrac{1}{4}\pi) + \lim_{u\to-\infty} (\tfrac{1}{4}\pi - \arctan e^u) \\
&= \tfrac{1}{2}\pi - \tfrac{1}{4}\pi + \tfrac{1}{4}\pi - 0 = \tfrac{1}{2}\pi
\end{aligned}$$

11. Evaluate $\int_0^{+\infty} e^{-x} \sin x\, dx$.

The upper limit of integration is infinite. We consider

$$\lim_{U\to+\infty}\int_0^U e^{-x}\sin x\,dx = \lim_{U\to+\infty}\left[-\tfrac{1}{2}e^{-x}(\sin x+\cos x)\right]_0^U = \lim_{U\to+\infty}\left[-\tfrac{1}{2}e^{-U}(\sin U+\cos U)\right]+\tfrac{1}{2}$$

As $U\to+\infty$, $e^{-U}\to 0$ while $\sin u$ and $\cos u$ vary from 1 to -1. Hence, $\int_0^{+\infty} e^{-x}\sin x\,dx = \frac{1}{2}$.

12. Find the area between the curve $y^2 = \dfrac{x^2}{1-x^2}$ and its asymptotes. (See Fig. 52-2.)

The required area is $A = 4\int_0^1 \dfrac{x\,dx}{\sqrt{1-x^2}}$, as can be seen from the approximating rectangle in the figure. Since the integrand is discontinuous at $x=1$, we consider

$$\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{x\,dx}{\sqrt{1-x^2}} = \lim_{\epsilon\to 0^+}\left[-(1-x^2)^{1/2}\right]_0^{1-\epsilon} = \lim_{\epsilon\to 0^+}\left(1-\sqrt{2\epsilon-\epsilon^2}\right) = 1$$

The required area is $4(1) = 4$ square units.

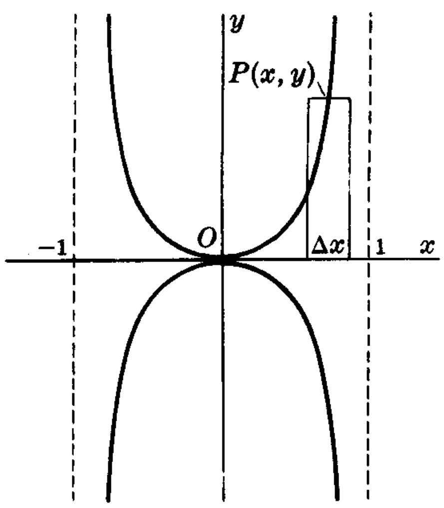

Fig. 52-2

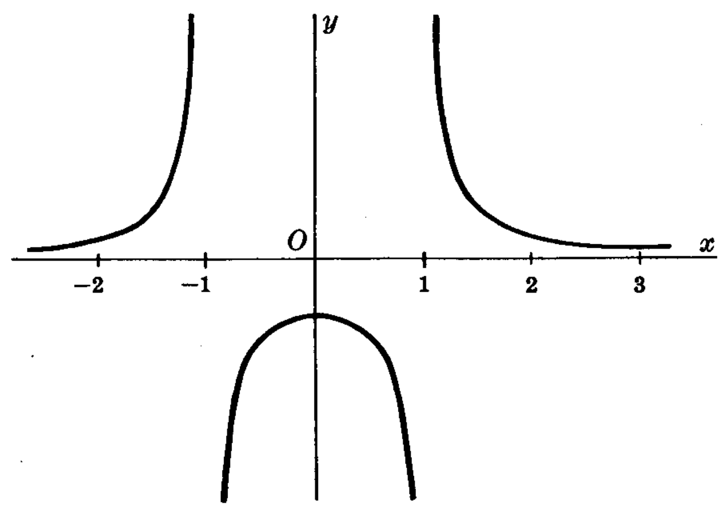

Fig. 52-3

13. Find the area lying to the right of $x=3$ and between the curve $y = \dfrac{1}{x^2-1}$ and the x axis. (See Fig. 52-3.)

$$A = \int_3^{+\infty}\frac{dx}{x^2-1} = \lim_{U\to+\infty}\int_3^U\frac{dx}{x^2-1} = \frac{1}{2}\lim_{U\to+\infty}\left[\ln\frac{x-1}{x+1}\right]_3^U = \frac{1}{2}\lim_{U\to+\infty}\ln\frac{U-1}{U+1} - \frac{1}{2}\ln\frac{1}{2}$$

$$= \frac{1}{2}\lim_{U\to+\infty}\ln\frac{1-1/U}{1+1/U} + \frac{1}{2}\ln 2 = \left(\frac{1}{2}\ln 2\right)\text{ square units}$$

Supplementary Problems

14. Evaluate the integral on the left in each of the following:

(a) $\displaystyle\int_0^1\frac{dx}{\sqrt{x}} = 2$ (b) $\displaystyle\int_0^4\frac{dx}{4-x}$ (meaningless) (c) $\displaystyle\int_0^4\frac{dx}{\sqrt{4-x}} = 4$

(*d*) $\int_0^4 \frac{dx}{(4-x)^{3/2}}$ (meaningless) (*e*) $\int_{-2}^{2} \frac{dx}{\sqrt{4-x^2}} = \pi$ (*f*) $\int_{-1}^{8} \frac{dx}{x^{1/3}} = \frac{9}{2}$

(*g*) $\int_0^4 \frac{dx}{(x-2)^{2/3}} = 6\sqrt[3]{2}$ (*h*) $\int_{-1}^{1} \frac{dx}{x^4}$ (meaningless) (*i*) $\int_0^1 \ln x\, dx = -1$

(*j*) $\int_0^1 x \ln x\, dx = -\frac{1}{4}$

15. Find the area between the given curve and its asymptotes. (Answers are in square units.)

(*a*) $y^2 = \frac{x^4}{4-x^2}$ (*b*) $y^2 = \frac{4-x}{x}$ (*c*) $y^2 = \frac{1}{x(1-x)}$ *Ans.* (*a*) 4π; (*b*) 4π; (*c*) 2π

16. Evaluate the integral on the left in each of the following:

(*a*) $\int_1^{+\infty} \frac{dx}{x^2} = 1$ (*b*) $\int_{-\infty}^{0} \frac{dx}{(4-x)^2} = \frac{1}{4}$ (*c*) $\int_0^{+\infty} e^{-x}\, dx = 1$

(*d*) $\int_{-\infty}^{6} \frac{dx}{(4-x)^2}$ (meaningless) (*e*) $\int_2^{+\infty} \frac{dx}{x \ln^2 x} = \frac{1}{\ln 2}$ (*f*) $\int_1^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx = \frac{2}{e}$

(*g*) $\int_{-\infty}^{+\infty} xe^{-x^2}\, dx = 0$ (*h*) $\int_{-\infty}^{+\infty} \frac{dx}{1+4x^2} = \frac{\pi}{2}$ (*i*) $\int_{-\infty}^{0} xe^x\, dx = -1$

(*j*) $\int_0^{+\infty} x^3 e^{-x}\, dx = 6$

17. Find the area between the given curve and its asymptote. (Answers are in square units.)

(*a*) $y = \frac{8}{x^2+4}$ (*b*) $y = \frac{x}{(4+x^2)^2}$ (*c*) $y = xe^{-x^2/2}$ *Ans.* (*a*) 4π; (*b*) $\frac{1}{4}$; (*c*) 2

18. Find the area (*a*) under $y = \frac{1}{x^2-4}$ and to the right of $x = 3$; (*b*) under $y = \frac{1}{x(x-1)^2}$ and to the right of $x = 2$.

Ans. (*a*) $\frac{1}{4}\ln 5$ square units; (*b*) $1 - \ln 2$ square units

19. Show that the following are meaningless: (*a*) the area under $y = \frac{1}{4-x^2}$ from $x = 2$ to $x = -2$; (*b*) the area under $xy = 9$ to the right of $x = 1$.

20. Show that the first-quadrant area under $y = e^{-2x}$ is $\frac{1}{2}$ square unit, and the volume generated by revolving the area about the x axis is $\frac{1}{4}\pi$ cubic units.

21. Show that when the portion R of the plane under $xy = 9$ and to the right of $x = 1$ is revolved about the x axis the volume generated is 81π cubic units but the area of the surface is infinite.

22. Find the length of the indicated arc:

(*a*) $9y^2 = x(3-x)^2$, a loop (*b*) $x^{2/3} + y^{2/3} = a^{2/3}$, entire length (*c*) $9y^2 = x^2(2x+3)$, a loop

Ans. (*a*) $4\sqrt{3}$ units; (*b*) $6a$ units; (*c*) $2\sqrt{3}$ units

23. Find the moment of inertia of a circle of radius r with respect to a tangent. *Ans.* $3r^2s/2$

24. Show that $\int_0^{+\infty} \frac{dx}{x^p}$ diverges for all values of p.

25. (*a*) Show that $\int_a^b \frac{dx}{(x-b)^p}$ exists for $p < 1$ and is meaningless for $p \geq 1$.

(*b*) Show that $\int_a^{+\infty} \frac{dx}{x^p}$ exists for $p > 1$ and is meaningless for $p \leq 1$.

26. Let $f(x) \le g(x)$ be defined and nonnegative everywhere on the interval $a \le x < b$, and let $\lim_{x\to b^-} f(x) = +\infty$ and $\lim_{x\to b^-} g(x) = +\infty$. From Fig. 52-4, it appears reasonable to assume that (1) if $\int_a^b g(x)\,dx$ exists so also does $\int_a^b f(x)\,dx$ and (2) if $\int_a^b f(x)\,dx$ does not exist neither does $\int_a^b g(x)\,dx$.

As an example, consider $\int_0^1 \frac{dx}{1-x^4}$. For $0 \le x < 1$, $1-x^4 = (1-x)(1+x)(1+x^2) < 4(1-x)$ and $\frac{1/4}{1-x} < \frac{1}{1-x^4}$. Since $\frac{1}{4}\int_0^1 \frac{dx}{1-x}$ does not exist, neither does the given integral.

Now consider $\int_0^1 \frac{dx}{x^2+\sqrt{x}}$. For $0 < x \le 1$, $\frac{1}{x^2+\sqrt{x}} < \frac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{dx}{\sqrt{x}}$ exists so also does the given integral.

Determine whether or not each of the following exists: (a) $\int_0^1 \frac{e^x\,dx}{x^{1/3}}$; (b) $\int_0^{\pi/4} \frac{\cos x}{x}\,dx$; (c) $\int_0^{\pi/4} \frac{\cos x}{\sqrt{x}}\,dx$.

Ans. (a) and (c) exist

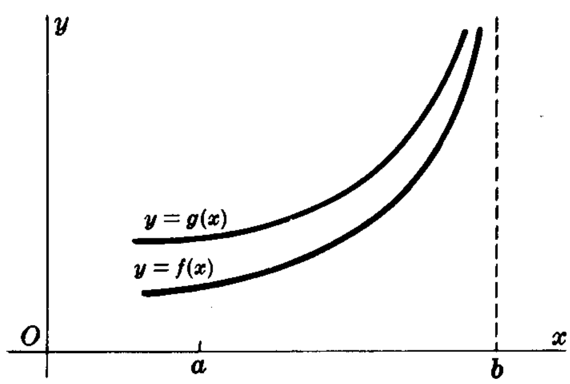

Fig. 52-4

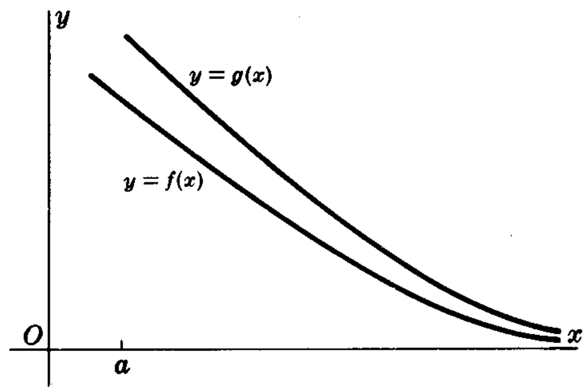

Fig. 52-5

27. Let $f(x) \le g(x)$ be defined and nonnegative everywhere on the interval $x \ge a$ while $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = 0$. From Fig. 52-5, it appears reasonable to assume that (1) if $\int_a^{+\infty} g(x)\,dx$ exists so also does $\int_a^{+\infty} f(x)\,dx$ and (2) if $\int_a^{+\infty} f(x)\,dx$ does not exist neither does $\int_a^{+\infty} g(x)\,dx$.

As an example, consider $\int_1^{+\infty} \frac{dx}{\sqrt{x^4+2x+6}}$. For $x \ge 1$, $\frac{1}{\sqrt{x^4+2x+6}} < \frac{1}{x^2}$. Since $\int_1^{+\infty} \frac{dx}{x^2}$ exists so also does the given integral.

Determine whether or not each of the following exists: (a) $\int_2^{+\infty} \frac{dx}{\sqrt{x^3+2x}}$; (b) $\int_1^{+\infty} e^{-x^2}\,dx$; (c) $\int_0^{+\infty} \frac{dx}{\sqrt{x+x^4}}$.

Ans. all exist

Chapter 53

Infinite Sequences and Series

AN INFINITE SEQUENCE $\{s_n\} = s_1, s_2, s_3, \ldots, s_n, \ldots$ is a function of n whose domain is the set of positive integers. (See Chapter 6.)

A sequence $\{s_n\}$ is said to be *bounded* if there exist numbers P and Q such that $P \le s_n \le Q$ for all values of n. For example, $\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots, \frac{2n+1}{2n}, \ldots$ is bounded since, for all n, $1 \le s_n \le 2$; but $2, 4, 6, \ldots, 2n, \ldots$ is not bounded.

A sequence $\{s_n\}$ is called *nondecreasing* if $s_1 \le s_2 \le s_3 \le \cdots \le s_n \le \cdots$, and is called *nonincreasing* if $s_1 \ge s_2 \ge s_3 \ge \cdots \ge s_n \ge \cdots$. For example, the sequences $\left\{\frac{n^2}{n+1}\right\} = \frac{1}{2}, \frac{4}{3}, \frac{9}{4}, \frac{16}{5}, \ldots$ and $\{2n - (-1)^n\} = 3, 3, 7, 7, \ldots$ are nondecreasing; and the sequences $\left\{\frac{1}{n}\right\} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ and $\{-n\} = -1, -2, -3, -4, \ldots$ are nonincreasing.

A sequence $\{s_n\}$ is said to converge to the finite number s as limit $\left(\lim_{n\to+\infty} s_n = s\right)$ if for any positive number ϵ, however small, there exists a positive integer m such that whenever $n > m$, then $|s - s_n| < \epsilon$. If a sequence has a limit, it is called a *convergent sequence*; otherwise, it is a *divergent sequence*. (See Problems 1 and 2.)

A sequence $\{s_n\}$ is said to diverge to ∞, and we write $\lim_{n\to+\infty} s_n = \infty$, if, for any positive number M, however large, there exists a positive integer m such that, whenever $n > m$, then $|s_n| > M$. If we replace $|s_n| > M$ in this definition by $s_n > M$, we obtain the definition of the expression $\lim_{n\to+\infty} s_n = +\infty$; and, if we replace $|s_n| > M$ by $s_n < -M$, we obtain the definition of $\lim_{n\to+\infty} s_n = -\infty$.

THEOREMS ON SEQUENCES

Theorem 53.1: Every bounded nondecreasing (nonincreasing) sequence is convergent.

A proof of this basic theorem is beyond the scope of this book.

Theorem 53.2: Every unbounded sequence is divergent.

(For a proof, see Problem 3.)

A number of the remaining theorems are merely restatements of those given in Chapter 7.

Theorem 53.3: A convergent (divergent) sequence remains convergent (divergent) after any or all of its first n terms are altered.

Theorem 53.4: The limit of a convergent sequence is unique.

(For a proof, see Problem 4.)

For Theorems 53.5 to 53.8, assume $\lim_{n\to+\infty} s_n = A$ and $\lim_{n\to+\infty} t_n = B$.

Theorem 53.5: $\lim_{n\to+\infty} (ks_n) = k \lim_{n\to+\infty} s_n = kA$, for k constant

Theorem 53.6: $\lim_{n\to+\infty} (s_n \pm t_n) = \lim_{n\to+\infty} s_n \pm \lim_{n\to+\infty} t_n = A \pm B$

Theorem 53.7: $\lim_{n\to+\infty} (s_n t_n) = \lim_{n\to+\infty} s_n \lim_{n\to+\infty} t_n = AB$

Theorem 53.8: $\lim_{n\to+\infty} \frac{s_n}{t_n} = \frac{\lim_{n\to+\infty} s_n}{\lim_{n\to+\infty} t_n} = \frac{A}{B}$, if $t \ne 0$ and $t_n \ne 0$ for all n

Theorem 53.9: If $\{s_n\}$ is a sequence of nonzero terms and if $\lim_{n\to+\infty} s_n = \infty$, then $\lim_{n\to+\infty} 1/s_n = 0$.

(For a proof, see Problem 5.)

Theorem 53.10: If $a > 1$, then $\lim_{n\to+\infty} a^n = +\infty$.

(For a proof, see Problem 6.)

Theorem 53.11: If $|r| < 1$, then $\lim_{n\to+\infty} r^n = 0$.

INFINITE SERIES. Let $\{s_n\}$ be an infinite sequence. By the *infinite series*

$$\sum s_n = \sum_{n=1}^{+\infty} s_n = s_1 + s_2 + s_3 + \cdots + s_n + \cdots \tag{53.1}$$

we mean the following sequence $\{S_n\}$ of *partial sums* S_n:

$$S_1 = s_1, \quad S_2 = s_1 + s_2, \quad S_3 = s_1 + s_2 + s_3, \ldots, \quad S_n = s_1 + s_2 + s_3 + \cdots + s_n, \quad \ldots$$

The numbers $s_1, s_2, s_3, \ldots$ are called the *terms* of the series $\Sigma\, s_n$.

If $\lim_{n\to+\infty} S_n = S$, a finite number, then the series (*53.1*) is said to *converge* and S is called its *sum*. If $\lim_{n\to+\infty} S_n$ does not exist, the series (*53.1*) is said to *diverge*. A series diverges either because $\lim_{n\to+\infty} S_n = \infty$ or because, as n increases, S_n increases and decreases without approaching a limit. An example of the latter is the *oscillating* series $1 - 1 + 1 - 1 \cdots$. Here, $S_1 = 1$, $S_2 = 0$, $S_3 = 1$, $S_4 = 0, \ldots$. (See Problems 7 and 8.)

From the theorems above, follow several more:

Theorem 53.12: A convergent (divergent) series remains convergent (divergent) after any or all of its first n terms are altered.

(See Problem 9.)

Theorem 53.13: The sum of a convergent series is unique.

Theorem 53.14: If $\Sigma\, s_n$ converges to S, then $\Sigma\, ks_n$, k being any constant, converges to kS. If $\Sigma\, s_n$ diverges, so also does $\Sigma\, ks_n$, if $k \neq 0$.

Theorem 53.15: If $\Sigma\, s_n$ converges, then $\lim_{n\to+\infty} s_n = 0$. (For a proof, see Problem 10.)

The converse is not true. For the harmonic series (Problem 7(*c*)), $\lim_{n\to+\infty} s_n = 0$ but the series diverges.

Theorem 53.16: If $\lim_{n\to+\infty} s_n \neq 0$, then $\Sigma\, s_n$ diverges. (See also Problem 11.)

The converse is not true; see Problem 7(*c*).

Let the sequence $\{s_n\}$ converge to s. Lay off on a number scale (Fig. 53-1) the points s, $s - \epsilon$, $s + \epsilon$, where ϵ is any small positive number. Now locate in order the points $s_1, s_2, s_3, \ldots$. The definition of convergence assures us that while the first m points may lie outside the ϵ-neighborhood of s, the point s_{m+1} and all subsequent points will lie within the neighborhood.

In Fig. 53-2, a rectangular coordinate system is used to illustrate the same idea. First draw in the lines $y = s$, $y = s - \epsilon$, and $y = s + \epsilon$, determining a band (shaded) of width 2ϵ. Now locate in turn the points $(1, s_1)$, $(2, s_2)$, $(3, s_3), \ldots$. As before, the point $(m + 1, s_{m+1})$ and all subsequent points lie within the band, for a suitably larger value of m.

It is important to note that only a finite number of points of a convergent sequence lie outside an ϵ-interval or ϵ-band.

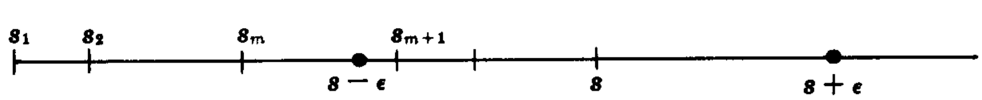

Fig. 53-1

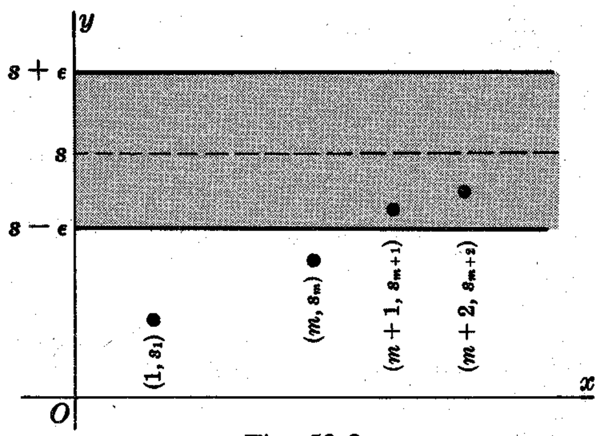

Fig. 53-2

Solved Problems

1. Use Theorem 53.1 to show that the sequences (a) $\left\{1-\frac{1}{n}\right\}$ and (b) $\left\{\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot4\cdot6\cdot8\cdots(2n)}\right\}$ are convergent.

(a) The sequence is bounded because $0 \le s_n \le 1$ for all n. Since $s_{n+1} = 1 - \frac{1}{n+1} = 1 - \frac{1}{n} + \frac{1}{n(n+1)} = s_n + \frac{1}{n(n+1)}$, that is $s_{n+1} \ge s_n$, the sequence is nondecreasing. Thus the sequence converges to some number $s \le 1$.

(b) The sequence is bounded because $0 \le s_n \le 1$ for every n. Since $s_{n+1} = \frac{1\cdot3\cdot5\cdot7\cdots(2n+1)}{2\cdot4\cdot6\cdot8\cdots(2n+2)} = \frac{2n+1}{2n+2}s_n$, the sequence is nonincreasing. Thus the sequence converges to some number $s \ge 0$.

2. Use Theorem 53.2 to show that the sequence $\left\{\frac{n!}{2^n}\right\}$ is divergent.

Since $\frac{n!}{2^n} = \frac{(1)(2)(3)\cdots(n)}{(2)(2)(2)\cdots(2)} = \frac{1}{2}\frac{3}{2}\frac{4}{2}\cdots\frac{n}{2} > \frac{n}{2}$ for $n > 4$, it follows that the terms of the sequence are unbounded. Hence, by Theorem 53.2, the sequence diverges. In fact, $\lim_{n\to+\infty} \frac{n!}{2^n} = +\infty$.

3. Prove: Every unbounded sequence $\{s_n\}$ is divergent.

Suppose $\{s_n\}$ were convergent. Then for any positive ϵ, however small, there would exist a positive integer m such that whenever $n > m$, then $|s_n - s| < \epsilon$. Since all but a finite number of the terms of the sequence would lie within this interval, the sequence would be bounded. But this is contrary to the hypothesis; hence the sequence is divergent.

4. Prove: The limit of a convergent sequence is unique.

Suppose the contrary, so that $\lim_{n\to+\infty} s_n = s$ and $\lim_{n\to+\infty} s_n = t$, where $|s-t| > 2\epsilon > 0$. Now the ϵ-neighborhoods of s and t have two contradictory properties: (1) they have no points in common, and (2) each contains all but a finite number of terms of the sequence. Thus $s = t$ and the limit is unique.

5. Prove: If $\{s_n\}$ is a sequence of nonzero terms and if $\lim_{n\to+\infty} s_n = \infty$, then $\lim_{n\to+\infty} 1/s_n = 0$.

Let $\epsilon > 0$ be chosen. From $\lim_{n\to+\infty} s_n = \infty$, it follows that for any $M > 1/\epsilon$, there exists a positive integer m such that whenever $n > m$ then $|s_n| > M > 1/\epsilon$. For this m, $|1/s_n| < 1/M < \epsilon$ whenever $n > m$; hence, $\lim_{n\to+\infty} 1/s_n = 0$.

6. Prove: If $a > 1$, then $\lim_{n\to+\infty} a^n = +\infty$.

Let $M > 0$ be chosen. Suppose $a = 1 + b$, for $b > 0$; then

$$a^n = (1+b)^n = 1 + nb + \frac{n(n-1)}{1(2)} b^2 + \cdots > 1 + nb > M$$

when $n > Mb$. Thus an effective m is the largest integer in M/b.

7. Prove:

(*a*) The infinite arithmetic series $a + (a+d) + (a+2d) + \cdots + [a + (n-1)d] + \cdots$ diverges when $a^2 + d^2 > 0$.

(*b*) The infinite geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$, where $a \neq 0$, converges to $\dfrac{a}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.

(*c*) The *harmonic* series $1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n + \cdots$ diverges.

(*a*) Here $S_n = \frac{1}{2}n[2a + (n-1)d]$ and $\lim_{n\to+\infty} S_n = \infty$ unless $a = d = 0$. Thus the series diverges when $a^2 + d^2 > 0$.

(*b*) Here $S_n = \dfrac{a - ar^n}{1-r} = \dfrac{a}{1-r} - \dfrac{a}{1-r} r^n$, $r \neq 1$. If $|r| < 1$, $\lim_{n\to+\infty} r^n = 0$, and $\lim_{n\to+\infty} S_n = \dfrac{a}{1-r}$.

If $|r| > 1$, $\lim_{n\to+\infty} r^n = \infty$, and S_n diverges.

If $|r| = 1$, the series is either $a + a + a + \cdots$ or $a - a + a - a + \cdots$ and diverges.

(*c*) When the partial sums are formed, it is found that $S_4 > 2$, $S_8 > 2.5$, $S_{16} > 3$, $S_{32} > 3.5$, $S_{64} > 4, \ldots$. Thus the sequence of partial sums (and hence the series) is unbounded and diverges.

8. Find S_n and S for (*a*) the series $\dfrac{1}{5} + \dfrac{1}{5^2} + \dfrac{1}{5^3} + \cdots$ and (*b*) the series $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots$.

(*a*)
$$S_1 = \frac{1}{5} = \frac{1}{4}\left(1 - \frac{1}{5}\right) \qquad S_2 = \frac{1}{5} + \frac{1}{5^2} = \frac{1}{4}\left(1 - \frac{1}{5^2}\right) \qquad S_3 = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} = \frac{1}{4}\left(1 - \frac{1}{5^3}\right) \quad \cdots$$

$$S_n = \frac{1}{4}\left(1 - \frac{1}{5^n}\right) \quad \text{and} \quad S = \lim_{n\to+\infty} \frac{1}{4}\left(1 - \frac{1}{5^n}\right) = \frac{1}{4}$$

(*b*)
$$S_1 = \frac{1}{1\cdot 2} = 1 - \frac{1}{2} \qquad S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = S_2 + \frac{1}{3\cdot 4} = 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \quad \cdots$$

$$S_n = 1 - \frac{1}{n+1} \quad \text{and} \quad S = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1$$

9. The series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges to 2. Examine the series that results when (*a*) its first four terms are dropped; (*b*) the terms $8 + 4 + 2$ are adjoined to the series.

(*a*) The series $\frac{1}{16} + \frac{1}{32} + \cdots$ is an infinite geometric series with $r = \frac{1}{2}$. It converges to $S = 2 - (1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = \frac{1}{8}$.

(*b*) The series $8 + 4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ is an infinite geometric series with $r = \frac{1}{2}$. It converges to $s = 2 + (8 + 4 + 2) = 16$.

10. Prove: If $\Sigma\, s_n = S$, then $\lim_{n\to+\infty} s_n = 0$.

Since $\Sigma\, s_n = S$, $\lim_{n\to+\infty} S_n = S$ and $\lim_{n\to+\infty} S_{n-1} = S$. Now $s_n = S_n - S_{n-1}$; hence,

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} (S_n - S_{n-1}) = \lim_{n\to+\infty} S_n - \lim_{n\to+\infty} S_{n-1} = S - S = 0$$

11. Show that the series (*a*) $\frac{1}{3}+\frac{2}{5}+\frac{3}{7}+\frac{4}{9}+\cdots$ and (*b*) $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$ diverge.

(*a*) Here $s_n=\dfrac{n}{2n+1}$ and $\lim\limits_{n\to+\infty} s_n=\lim\limits_{n\to+\infty}\dfrac{n}{2n+1}=\lim\limits_{n\to+\infty}\dfrac{1}{2+1/n}=\dfrac{1}{2}\neq 0$.

(*b*) Here $s_n=\dfrac{2^n-1}{2^n}$ and $\lim\limits_{n\to+\infty}\dfrac{2^n-1}{2^n}=\lim\limits_{n\to+\infty}\left(1-\dfrac{1}{2^n}\right)=1\neq 0$.

12. A series $\Sigma\, s_n$ converges to S as limit if the sequence $\{S_n\}$ of partial sums converges to S, that is, if for any $\epsilon>0$, however small, there exists an integer m such that whenever $n>m$ then $|S-S_n|<\epsilon$. Show that the series of Problem 8 converge by producing for each an effective m for any given ϵ.

(*a*) If $|S-S_n|=\left|\dfrac{1}{4}-\dfrac{1}{4}\left(1-\dfrac{1}{5^n}\right)\right|=\dfrac{1}{4\cdot 5^n}<\epsilon$, then $5^n>\dfrac{1}{4\epsilon}$, $n\ln 5>-\ln(4\epsilon)$, and $n>-\dfrac{\ln 4\epsilon}{\ln 5}$. Thus, $m=$ greatest integer not greater than $-\dfrac{\ln 4\epsilon}{\ln 5}$ is effective.

(*b*) If $|S-S_n|=\left|1-\left(1-\dfrac{1}{n+1}\right)\right|=\dfrac{1}{n+1}<\epsilon$, then $n+1>\dfrac{1}{\epsilon}$ and $n>\dfrac{1}{\epsilon}-1$. Thus, $m=$ greatest integer not greater than $\dfrac{1}{\epsilon}-1$ is effective.

Supplementary Problems

13. Determine for each sequence whether or not it is bounded, nonincreasing or nondecreasing, and convergent or divergent.

(*a*) $\left\{n+\dfrac{2}{n}\right\}$ (*b*) $\left\{\dfrac{(-1)^n}{n}\right\}$ (*c*) $\{\sin\frac{1}{4}n\pi\}$ (*d*) $\{\sqrt[3]{n^2}\}$ (*e*) $\left\{\dfrac{n!}{10^n}\right\}$ (*f*) $\left\{\dfrac{\ln n}{n}\right\}$

Ans. (*a*), (*d*), and (*e*) are unbounded; (*a*), (*d*), and (*e*) are nondecreasing, (*f*) is nonincreasing, and (*b*) and (*c*) are neither nonincreasing nor nondecreasing; (*b*) and (*f*) are convergent

14. Show that $\lim\limits_{n\to+\infty}\sqrt[n]{1/n^p}=1$, for $p>0$. (*Hint*: $n^{p/n}=e^{(p\ln n)/n}$.)

15. For the sequence $\left\{\dfrac{n}{n+1}\right\}$, verify that (*a*) the neighborhood $|1-s_n|<0.01$ contains all but the first 99 terms of the sequence, (*b*) the sequence is bounded, and (*c*) $\lim\limits_{n\to+\infty} s_n=1$.

16. Prove: If $|r|<1$, then $\lim\limits_{n\to+\infty} r^n=0$.

17. Examine each of the following geometric series for convergence. If the series converges, find its sum.
(*a*) $1+1/2+1/4+1/8+\cdots$ (*b*) $4-1+1/4-1/16+\cdots$ (*c*) $1+3/2+9/4+27/8+\cdots$

Ans. (*a*) $S=2$; (*b*) $S=\frac{16}{5}$; (*c*) diverges

18. Find the sum of each of the following series.

(*a*) $\sum 3^{-n}$ (*b*) $\sum\dfrac{1}{(2n-1)(2n+1)}$ (*c*) $\sum\left(\dfrac{1}{n^p}-\dfrac{1}{(n+1)^p}\right)$, $p>0$

(*d*) $\sum\dfrac{1}{n(n+2)}$ (*e*) $\sum\dfrac{1}{n(n+3)}$ (*f*) $\sum\dfrac{n}{(n+1)!}$

(*g*) $\sum\dfrac{1}{(4n-3)(4n+1)}$ (*h*) $\sum\dfrac{1}{n(n+1)(n+2)}$

Ans. (*a*) $\frac{1}{2}$; (*b*) $\frac{1}{2}$; (*c*) 1; (*d*) $\frac{3}{4}$; (*e*) $\frac{11}{18}$; (*f*) 1; (*g*) $\frac{1}{4}$; (*h*) $\frac{1}{4}$

19. Show that each of the following diverges.

(*a*) $3 + 5/2 + 7/3 + 9/4 + \cdots$ (*b*) $2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$

(*c*) $e + e^2/8 + e^3/27 + e^4/64 + \cdots$ (*d*) $\sum \dfrac{1}{\sqrt{n} + \sqrt{n-1}}$

20. Prove: If $\lim_{n\to+\infty} s_n \neq 0$, then $\Sigma\, s_n$ diverges.

21. Prove that the series of Problem 18(*a*) to (*d*) converge by producing for each an effective positive integer m such that for $\epsilon > 0$, $|S - S_n| < \epsilon$ whenever $n > m$.

Ans. m = greatest integer not greater than (*a*) $-\dfrac{\ln 2\epsilon}{\ln 3}$; (*b*) $\dfrac{1}{4\epsilon} - \dfrac{1}{2}$; (*c*) $\sqrt[p]{1/\epsilon} - 1$; (*d*) the positive root of $2\epsilon m^2 - 2(1 - 3\epsilon)m - (3 - 4\epsilon) = 0$

Chapter 54

Tests for the Convergence and Divergence of Positive Series

SERIES OF POSITIVE TERMS. A series $\Sigma\, s_n$, all of whose terms are positive, is called a *positive* series.

Theorem 54.1: A positive series $\Sigma\, s_n$ is convergent if the sequence of partial sums $\{S_n\}$ is bounded.

This theorem follows from the fact that the sequence of partial sums of a positive series is always nondecreasing.

Theorem 54.2 (the integral test): Let $f(x)$ be a function such that $f(n)$ is the general term s_n of the series $\Sigma\, s_n$ of positive terms. If $f(x) > 0$ and never increases on the interval $x > \xi$, where ξ is some positive integer, then the series $\Sigma\, s_n$ converges or diverges according as $\int_{\xi}^{+\infty} f(x)\,dx$ exists or does not exist.

(See Problems 1 to 5.)

Theorem 54.3 (the comparison test for convergence): A positive series $\Sigma\, s_n$ is convergent if each term (perhaps, after a finite number) is less than or equal to the corresponding term of a known convergent positive series $\Sigma\, c_n$.

Theorem 54.4 (the comparison test for divergence): A positive series $\Sigma\, s_n$ is divergent if each term (perhaps, after a finite number) is equal to or greater than the corresponding term of a known divergent positive series $\Sigma\, d_n$.

(See Problems 6 to 11.)

Theorem 54.5 (the ratio test): A positive series $\Sigma\, s_n$ converges if $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} < 1$, and diverges if $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} > 1$ or if $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = +\infty$. If $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = 1$ or if the limit does not exist, the test gives no information about convergence or divergence.

(See Problems 12 to 18.)

Solved Problems

THE INTEGRAL TEST

1. Prove the integral test: Let $f(n)$ denote the general term s_n of the positive series $\Sigma\, s_n$. If $f(x) > 0$ and never increases on the interval $x > \xi$, where ξ is a positive integer, then the series $\Sigma\, s_n$ converges or diverges according as $\int_{\xi}^{+\infty} f(x)\,dx$ exists or does not exist.

In Fig. 54-1, the area under the curve $y = f(x)$ from $x = \xi$ to $x = n$ has been approximated by two sets of rectangles having unit bases. Expressing the fact that the area under the curve lies between the sum of the areas of the small rectangles and the sum of the areas of the large rectangles, we have

$$s_{\xi+1} + s_{\xi+2} + \cdots + s_n < \int_{\xi}^{n} f(x)\,dx < s_{\xi} + s_{\xi+1} + \cdots + s_{n-1}$$

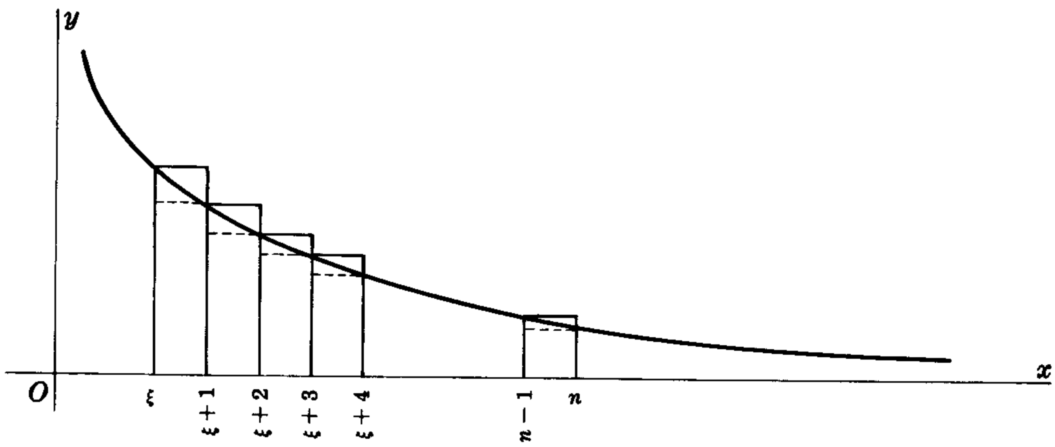

Fig. 54-1

Suppose $\lim_{n\to+\infty}\int_{\xi}^{n} f(x)\,dx = \int_{\xi}^{+\infty} f(x)\,dx = A$. Then

$$s_{\xi+1} + s_{\xi+2} + \cdots + s_n < A$$

and $S_n = s_\xi + s_{\xi+1} + \cdots + s_n$ is bounded and nondecreasing, as n increases. Thus, by Theorem 54.1, $\Sigma\, s_n$ converges.

Now suppose $\lim_{n\to+\infty}\int_{\xi}^{n} f(x)\,dx = \int_{\xi}^{+\infty} f(x)\,dx$ does not exist. Then $S_n = s_\xi + s_{\xi+1} + \cdots + s_n$ is unbounded and $\Sigma\, s_n$ diverges.

In Problems 2 to 5, examine the series for convergence, using the integral test.

2. $\dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{7}} + \dfrac{1}{\sqrt{9}} + \cdots$

Here $f(n) = s_n = \dfrac{1}{\sqrt{2n+1}}$, so take $f(x) = \dfrac{1}{\sqrt{2x+1}}$. On the interval $x > 1$, $f(x) > 0$ and decreases as x increases. Take $\xi = 1$ and consider

$$\int_1^{+\infty} f(x)\,dx = \lim_{U\to+\infty}\int_1^U \frac{dx}{\sqrt{2x+1}} = \lim_{U\to+\infty}\,[\sqrt{2x+1}]_1^U = \lim_{U\to+\infty}\sqrt{2U+1} - \sqrt{3} = \infty$$

The integral does not exist, so the series is divergent.

3. $\dfrac{1}{4} + \dfrac{1}{16} + \dfrac{1}{36} + \dfrac{1}{64} + \cdots$

Here $f(n) = s_n = \dfrac{1}{4n^2}$, so we take $f(x) = \dfrac{1}{4x^2}$. On the interval $x > 1$, $f(x) > 0$ and decreases as x increases. We take $\xi = 1$ and consider

$$\int_1^{+\infty} f(x)\,dx = \frac{1}{4}\lim_{U\to+\infty}\int_1^U \frac{dx}{x^2} = \frac{1}{4}\lim_{U\to+\infty}\left[-\frac{1}{x}\right]_1^U = \frac{1}{4}\lim_{U\to+\infty}\left(-\frac{1}{U}+1\right) = \frac{1}{4}$$

The integral exists, and the series is convergent.

4. $\sin \pi + \frac{1}{4}\sin\frac{1}{2}\pi + \frac{1}{9}\sin\frac{1}{3}\pi + \frac{1}{16}\sin\frac{1}{4}\pi + \cdots$

Here $f(n) = s_n = \dfrac{1}{n^2}\sin\dfrac{1}{n}\pi$; we take $f(x) = \dfrac{1}{x^2}\sin\dfrac{1}{x}\pi$. On the interval $x > 2$, $f(x) > 0$ and decreases as x increases. We take $\xi = 2$ and consider

$$\int_2^{+\infty} f(x)\,dx = \lim_{U\to+\infty}\int_2^U \frac{1}{x^2}\sin\frac{1}{x}\pi\,dx = \frac{1}{\pi}\lim_{U\to+\infty}\left[\cos\frac{1}{x}\pi\right]_2^U = \frac{1}{\pi}$$

The series converges.

5. $1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$, for $p>0$ (the p series).

Here $f(n)=s_n=\frac{1}{n^p}$; take $f(x)=\frac{1}{x^p}$. On the interval $x>1$, $f(x)>0$ and decreases as x increases. Take $\xi=1$ and consider

$$\int_1^{+\infty} f(x)\,dx = \lim_{U\to+\infty}\int_1^U \frac{dx}{x^p} = \lim_{U\to+\infty}\left[\frac{x^{1-p}}{1-p}\right]_1^U = \frac{1}{1-p}\left(\lim_{U\to+\infty} U^{1-p}-1\right) \quad \text{for } p\neq 1$$

If $p>1$, $\frac{1}{1-p}\left(\lim_{U\to+\infty} U^{1-p}-1\right)=\frac{1}{1-p}\left(\lim_{U\to+\infty}\frac{1}{U^{p-1}}-1\right)=\frac{1}{p-1}$ and the series converges.

If $p=1$, $\int_1^{+\infty} f(x)\,dx=\lim_{U\to+\infty}\ln U=+\infty$ and the series diverges.

If $p<1$, $\frac{1}{1-p}\left(\lim_{U\to+\infty} U^{1-p}-1\right)=+\infty$ and the series diverges.

THE COMPARISON TEST

The general term of a series that is being tested for convergence is compared with general terms of known convergent and divergent series. The following series are useful as test series:

1. The geometric series $a+ar+ar^2+\cdots+ar^n+\cdots$, for $a\neq 0$, which converges for $0<r<1$ and diverges for $r\geq 1$
2. The p series $1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots+\frac{1}{n^p}+\cdots$, which converges for $p>1$ and diverges for $p\leq 1$
3. Each new series tested

In Problems 6 to 11, examine the series for convergence, using the comparison test.

6. $\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17}+\cdots+\frac{1}{n^2+1}+\cdots$

The general term of the series is $s_n=\frac{1}{n^2+1}<\frac{1}{n^2}$; hence the given series is term by term less than the p series $1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}+\cdots$. The test series is convergent because $p=2$, and so also is the given series. (The integral test may be used here as well.)

7. $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$

The general term of the series is $\frac{1}{\sqrt{n}}$. Since $\frac{1}{\sqrt{n}}\geq\frac{1}{n}$, the given series is term by term greater than or equal to the harmonic series and is divergent. (The integral test may be used here as well.)

8. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

The general term of the series is $\frac{1}{n!}$. Since $n!\geq 2^{n-1}$, $\frac{1}{n!}\leq\frac{1}{2^{n-1}}$. The given series is term by term less than or equal to the convergent geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ and is convergent. (The integral test cannot be used here.)

9. $2+\frac{3}{2^3}+\frac{4}{3^3}+\frac{5}{4^3}+\cdots$

The general term of the series is $\frac{n+1}{n^3}$. Since $\frac{n+1}{n^3}\leq\frac{2n}{n^3}=\frac{2}{n^2}$, the given series is term by term less than or equal to twice the convergent p series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ and is convergent.

10. $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$

The general term of the series is $\frac{1}{n^n}$. Since $\frac{1}{n^n} \le \frac{1}{2^{n-1}}$, the given series is term by term less than or equal to the convergent geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ and is convergent. (Also, the given series is term by term less than or equal to the convergent p series with $p = 2$.)

11. $1 + \frac{2^2+1}{2^3+1} + \frac{3^2+1}{3^3+1} + \frac{4^2+1}{4^3+1} + \cdots$

The general term is $\frac{n^2+1}{n^3+1} \ge \frac{1}{n}$. Hence the given series is term by term greater than or equal to the harmonic series and is divergent.

THE RATIO TEST

12. Prove the ratio test: A positive series $\Sigma\, s_n$ converges if $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} < 1$ and diverges if $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} > 1$.

Suppose $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = L < 1$. Then for any r, where $L < r < 1$, there exists a positive integer m such that whenever $n > m$ then $\frac{s_{n+1}}{s_n} < r$, that is,

$$\frac{s_{m+2}}{s_{m+1}} < r \quad \text{or} \quad s_{m+2} < rs_{m+1}$$

$$\frac{s_{m+3}}{s_{m+2}} < r \quad \text{or} \quad s_{m+3} < rs_{m+2} < r^2 s_{m+1}$$

$$\frac{s_{m+4}}{s_{m+3}} < r \quad \text{or} \quad s_{m+4} < rs_{m+3} < r^3 s_{m+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus each term of the series $s_{m+1} + s_{m+2} + s_{m+3} + \cdots$ is less than or equal to the corresponding term of the geometric series $s_{m+1} + rs_{m+1} + r^2 s_{m+1} + \cdots$ which converges since $r < 1$. Hence $\Sigma\, s_n$ is convergent by Theorem 54.3.

Suppose $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = L > 1$ (or $= +\infty$). Then there exists a positive integer m such that whenever $n > m$, $\frac{s_{n+1}}{s_n} > 1$. Now $s_{n+1} > s_n$, and $\{s_n\}$ does not converge to 0. Hence $\Sigma\, s_n$ diverges by Theorem 53.16.

Suppose $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = 1$. An example is the p series $\sum \frac{1}{n^p}$, $p > 0$, for which

$$\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to+\infty} \frac{n^p}{(n+1)^p} = \lim_{n\to+\infty} \left(\frac{1}{1+1/n}\right)^p = 1$$

Since the series converges when $p > 1$ and diverges when $p \le 1$, the test fails to indicate convergence or divergence.

In Problems 13 to 23, examine the series for convergence, using the ratio test.

13. $\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$

Here $s_n = \frac{n}{3^n}$, $s_{n+1} = \frac{n+1}{3^{n+1}}$, and $\frac{s_{n+1}}{s_n} = \frac{n+1}{3^{n+1}} \frac{3^n}{n} = \frac{n+1}{3n}$. Then $\lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = \lim_{n\to+\infty} \frac{n+1}{3n} = \frac{1}{3}$ and the series converges.

14. $\frac{1}{3}+\frac{2!}{3^2}+\frac{3!}{3^3}+\frac{4!}{3^4}+\cdots$

Here $s_n=\frac{n!}{3n}$, $s_{n+1}=\frac{(n+1)!}{3^{n+1}}$, and $\frac{s_{n+1}}{s_n}=\frac{n+1}{3}$. Then $\lim_{n\to+\infty}\frac{s_{n+1}}{s_n}=\lim_{n\to+\infty}\frac{n+1}{3}=\infty$ and the series diverges.

15. $1+\frac{1\cdot 2}{1\cdot 3}+\frac{1\cdot 2\cdot 3}{1\cdot 3\cdot 5}+\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 3\cdot 5\cdot 7}+\cdots$

Here $s_n=\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$, $s_{n+1}=\frac{(n+1)!}{1\cdot 3\cdot 5\cdots(2n+1)}$, and $\frac{s_{n+1}}{s_n}=\frac{n+1}{2n+1}$. Then $\lim_{n\to+\infty}\frac{n+1}{2n+1}=\frac{1}{2}$ and the series converges.

16. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$

Here $s_n=\frac{1}{(n)(2^n)}$, $s_{n+1}=\frac{1}{(n+1)(2^{n+1})}$, and $\frac{s_{n+1}}{s_n}=\frac{n}{2(n+1)}$. Then $\lim_{n\to+\infty}\frac{n}{2(n+1)}=\frac{1}{2}$ and the series converges.

17. $2+\frac{3}{2}\frac{1}{4}+\frac{4}{3}\frac{1}{4^2}+\frac{5}{4}\frac{1}{4^3}+\cdots$

Here $s_n=\frac{n+1}{n}\frac{1}{4^{n-1}}$, $s_{n+1}=\frac{n+2}{n+1}\frac{1}{4^n}$, and $\frac{s_{n+1}}{s_n}=\frac{n(n+2)}{4(n+1)^2}$. Then $\lim_{n\to+\infty}\frac{n(n+2)}{4(n+1)^2}=\frac{1}{4}$ and the series converges.

18. $1+\frac{2^2+1}{2^3+1}+\frac{3^s+1}{3^3+1}+\frac{4^2+1}{4^3+1}+\cdots$

$$s_n=\frac{n^2+1}{n^3+1}\qquad s_{n+1}=\frac{(n+1)^2+1}{(n+1)^3+1}\qquad \frac{s_{n+1}}{s_n}=\frac{(n+1)^2+1}{(n+1)^3+1}\,\frac{n^3+1}{n^2+1}$$

Then $\lim_{n\to+\infty}\frac{s_{n+1}}{s_n}=1$ and the test fails. (See Problem 12.)

Supplementary Problems

19. Verify that the integral test may be applied, and use the test to determine convergence or divergence:

(*a*) $\sum\frac{1}{n}$ (*b*) $\sum\frac{50}{n(n+1)}$ (*c*) $\sum\frac{1}{n\ln n}$ (*d*) $\sum\frac{n}{(n+1)(n+2)}$

(*e*) $\sum\frac{n}{n^2+1}$ (*f*) $\sum\frac{n}{e^n}$ (*g*) $\sum\frac{2n}{(n+1)(n+2)(n+3)}$ (*h*) $\sum\frac{1}{(2n+1)^2}$

Ans. (*a*), (*c*), (*d*), (*e*) divergent

20. Determine the convergence or divergence of each series, using the comparison test:

(a) $\sum \frac{1}{n^3-1}$ (b) $\sum \frac{n-2}{n^3}$ (c) $\sum \frac{1}{\sqrt[3]{n}}$ (d) $\sum \frac{1}{n^2+5}$

(e) $\sum \frac{n+2}{n(n+1)}$ (f) $\sum \frac{1}{n^{n-1}}$ (g) $\sum \frac{1}{3n+1}$ (h) $\sum \frac{\ln n}{n}$

(i) $\sum \frac{1}{3^n+1}$ (j) $\sum \frac{\ln n}{\sqrt{n}}$ (k) $\sum \frac{1}{3^n-1}$ (l) $\sum \frac{\ln n}{n^p}$

(m) $\sum \frac{n}{3n^2-4}$ (n) $\sum \frac{1}{1+\ln n}$ (o) $\sum \frac{n^4+5}{n^5}$ (p) $\sum \frac{n+1}{n\sqrt{3n-2}}$

Ans. (a), (b), (d), (f), (i), (k), (l) for $p>2$ convergent

21. Determine the convergence or divergence of each series, using the ratio test:

(a) $\sum \frac{(n+1)(n+2)}{n!}$ (b) $\sum \frac{5^n}{n!}$ (c) $\sum \frac{n}{2^{2n}}$ (d) $\sum \frac{3^{2n-1}}{n^2+n}$

(e) $\sum \frac{(n+1)2^n}{n!}$ (f) $\sum n\left(\frac{3}{4}\right)^n$ (g) $\sum \frac{n^3}{(\ln 2)^n}$ (h) $\sum \frac{n^3}{(\ln 3)^n}$

(i) $\sum \frac{2^n}{n(n+2)}$ (j) $\sum \frac{n^n}{n!}$ (k) $\sum \frac{2^n}{2n-1}$ (l) $\sum \frac{n^3}{3^n}$

Ans. (a), (b), (c), (e), (f), (h), (l) convergent

22. Determine the convergence or divergence of each series:

(a) $\frac{1}{4^2}+\frac{1}{7^2}+\frac{1}{10^2}+\frac{1}{13^2}+\cdots$ (b) $3+\frac{3}{\sqrt[3]{2}}+\frac{3}{\sqrt[3]{3}}+\frac{3}{\sqrt[3]{4}}+\cdots$

(c) $1+\frac{1}{5}+\frac{1}{9}+\frac{1}{13}+\cdots$ (d) $\frac{1}{2}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5\cdot 6}+\frac{1}{5\cdot 6\cdot 7\cdot 8}+\cdots$

(e) $3+\frac{3}{4}+\frac{11}{27}+\frac{9}{32}+\cdots$ (f) $\frac{2}{3}+\frac{3}{2\cdot 3^2}+\frac{4}{3\cdot 3^3}+\frac{5}{4\cdot 3^4}+\cdots$

(g) $\frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\cdots$ (h) $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\frac{4}{3\cdot 5}+\frac{5}{4\cdot 6}+\cdots$

(i) $\frac{1}{2}+\frac{2}{3^2}+\frac{3}{4^3}+\frac{4}{5^4}+\cdots$ (j) $1+\frac{1}{2^2}+\frac{1}{3^{5/2}}+\frac{1}{4^3}+\cdots$

(k) $2+\frac{3}{5}+\frac{4}{10}+\frac{5}{17}+\cdots$ (l) $\frac{2}{5}+\frac{2\cdot 4}{5\cdot 8}+\frac{2\cdot 4\cdot 6}{5\cdot 8\cdot 11}+\frac{2\cdot 4\cdot 6\cdot 8}{5\cdot 8\cdot 11\cdot 14}+\cdots$

Ans. (a), (d), (f), (g), (i), (j), (l) convergent

23. Prove the comparison test for convergence. (*Hint*: If $\Sigma\, c_n = C$, then $\{S_n\}$ is bounded.)

24. Prove the comparison test for divergence. $\left(\textit{Hint}: \sum_1^n s_i \ge \sum_1^n d_i > M \text{ for } n > m.\right)$

25. Prove the *polynomial test*: If $P(n)$ and $Q(n)$ are polynomials of degree p and q, respectively, the series $\sum \frac{P(n)}{Q(n)}$ converges if $q > p+1$ and diverges if $q \le p+1$. (*Hint*: Compare with $1/n^{q-p}$.)

26. Use the polynomial test to determine the convergence or divergence of each series:

(a) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$ (b) $\frac{1}{2}+\frac{1}{7}+\frac{1}{12}+\frac{1}{17}+\cdots$

(c) $\frac{3}{2}+\frac{5}{10}+\frac{7}{30}+\frac{9}{68}+\cdots$ (d) $\frac{3}{2}+\frac{5}{24}+\frac{7}{108}+\frac{9}{320}+\cdots$

(*e*) $\frac{1}{2^2-1}+\frac{2}{3^2-2}+\frac{3}{4^2-3}+\frac{4}{5^2-4}+\cdots$ (*f*) $\frac{1}{2^3-1^2}+\frac{1}{3^2-2^2}+\frac{1}{4^3-3^2}+\frac{1}{5^3-4^2}+\cdots$

(*g*) $\frac{2}{1\cdot 3}+\frac{3}{2\cdot 4}+\frac{4}{3\cdot 5}+\frac{5}{4\cdot 6}+\cdots$ *Ans.* (*a*), (*c*), (*d*), (*f*) convergent

27. Prove the *root test*: A positive series $\Sigma\, s_n$ converges if $\lim_{n\to+\infty} \sqrt[n]{s_n} < 1$ and diverges if $\lim_{n\to+\infty} \sqrt[n]{s_n} > 1$. The test fails if $\lim_{n\to+\infty} \sqrt[n]{s_n} = 1$. (*Hint*: If $\lim_{n\to+\infty} \sqrt[n]{s_n} < 1$, then $\sqrt[n]{s_n} < r < 1$ for $n > m$, and $s_n < r^n$.)

28. Use the root test to determine the convergence or divergence of (*a*) $\sum \frac{1}{n^n}$; (*b*) $\sum \frac{1}{(\ln n)^n}$; (*c*) $\sum \frac{2^n-1}{n^n}$; (*d*) $\sum \left(\frac{n}{n^2+2}\right)^n$. *Ans.* all convergent

Chapter 55

Series with Negative Terms

A SERIES having only negative terms may be treated as the negative of a positive series.

ALTERNATING SERIES. A series whose terms are alternately positive and negative, as

$$\sum (-1)^{n-1} s_n = s_1 - s_2 + s_3 - s_4 + \cdots + (-1)^{n-1} s_n \cdots \qquad (55.1)$$

in which each s_i is *positive*, is called an *alternating series*.

Theorem 55.1: An alternating series (*55.1*) converges if (1) $s_n > s_{n+1}$ for every value of n, and (2) $\lim_{n\to+\infty} s_n = 0$.

(See Problems 1 and 2.)

ABSOLUTE CONVERGENCE. A series $\Sigma\, s_n = s_1 + s_2 + \cdots + s_n + \cdots$, with mixed (positive and negative) terms, is called *absolutely convergent* if $\Sigma\, |s_n| = |s_1| + |s_2| + |s_3| + \cdots + |s_n| + \cdots$ converges.

Every convergent positive series is absolutely convergent. Every absolutely convergent series is convergent. (For a proof, see Problem 3.)

CONDITIONAL CONVERGENCE. If $\Sigma\, s_n$ converges while $\Sigma\, |s_n|$ diverges, $\Sigma\, s_n$ is called *conditionally convergent.*

As an example, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$ is conditionally convergent since it converges while $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

RATIO TEST FOR ABSOLUTE CONVERGENCE. A series $\Sigma\, s_n$ with mixed terms is absolutely convergent if $\lim_{n\to+\infty} \left|\frac{s_{n+1}}{s_n}\right| < 1$ and is divergent if $\lim_{n\to+\infty} \left|\frac{s_{n+1}}{s_n}\right| > 1$. If the limit is 1, the test gives no information. (See Problems 4 to 12.)

Solved Problems

1. Prove: An alternating series $s_1 - s_2 + s_3 - s_4 \cdots$ converges if (1) $s_n > s_{n+1}$ for every value of n, and (2) $\lim_{n\to+\infty} s_n = 0$.

Consider the partial sum $S_{2m} = s_1 - s_2 + s_3 - s_4 \cdots + s_{2m-1} - s_{2m}$, which may be grouped as follows:

$$S_{2m} = (s_1 - s_2) + (s_3 - s_4) + \cdots + (s_{2m-1} - s_{2m}) \qquad (1)$$

or

$$S_{2m} = s_1 - (s_2 - s_3) - \cdots - (s_{2m-2} - s_{2m-1}) - s_{2m} \qquad (2)$$

By hypothesis, $s_n > s_{n+1}$ so that $s_n - s_{n+1} > 0$. Hence, by (*1*), $0 < S_{2m} < S_{2m+2}$ and, by (*2*), $S_{2m} < s_1$. Thus, the sequence $\{S_{2m}\}$ is increasing and bounded and, therefore, converges to a limit $L < s_1$.

Consider next the partial sum $S_{2m+1} = S_{2m} + s_{2m+1}$; we have

$$\lim_{m\to+\infty} S_{2m+1} = \lim_{m\to+\infty} S_{2m} + \lim_{m\to+\infty} s_{2m+1} = L + 0 = L$$

Thus $\lim_{n\to+\infty} S_n = L$ and the series converges.

2. Show that the following alternating series converge.

(a) $1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} \cdots$:

$s_n = \dfrac{1}{n^2}$ and $s_{n+1} = \dfrac{1}{(n+1)^2}$; then $s_n > s_{n+1}$, $\lim_{n\to+\infty} s_n = 0$, and the series converges.

(b) $\frac{1}{2} - \frac{1}{5} + \frac{1}{10} - \frac{1}{17} \cdots$:

$s_n = \dfrac{1}{n^2+1}$ and $s_{n+1} = \dfrac{1}{(n+1)^2+1}$; then $s_n > s_{n+1}$, $\lim_{n\to+\infty} \dfrac{1}{n^2+1} = 0$, and the series converges.

(c) $\dfrac{1}{e} - \dfrac{2}{e^2} + \dfrac{3}{e^3} - \dfrac{4}{e^4} \cdots$:

The series converges since $s_n > s_{n+1}$ and $\lim_{n\to+\infty} \dfrac{n}{e^n} = \lim_{n\to+\infty} \dfrac{1}{e^n} = 0$, by l'Hospital's rule.

3. Prove: Every absolutely convergent series is convergent.

Let
$$\sum s_n = s_1 + s_2 + s_3 + s_4 + \cdots + s_n + \cdots$$
having both positive and negative terms, be the given series whose corresponding convergent positive series is

$$\sum |s_n| = |s_1| + |s_2| + |s_3| + \cdots + |s_n| + \cdots$$

For all n, $0 \le s_n + |s_n| \le 2|s_n|$. Since $\Sigma\,|s_n|$ converges, so does $\Sigma\, 2|s_n|$. By the comparison test, $\Sigma\,(s_n + |s_n|)$ also converges. Hence, $\Sigma\, s_n = \Sigma\,(s_n + |s_n|) - \Sigma\,|s_n|$ converges, since the difference of two convergent series is convergent.

ABSOLUTE AND CONDITIONAL CONVERGENCE

In Problems 4 to 12, examine the convergent series for absolute or conditional convergence.

4. $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} \cdots$

The series $1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \cdots$, obtained by making all the terms positive, is convergent, being a geometric series with $r = \frac{1}{2}$. Thus the given series is absolutely convergent.

5. $1 - \dfrac{2}{3} + \dfrac{3}{3^2} - \dfrac{4}{3^3} \cdots$

The series $1 + \dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \cdots$, obtained by making all the terms positive, is convergent by the ratio test. Thus the given series is absolutely convergent.

6. $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} \cdots$

The series $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \cdots$ diverges, being a p series with $p = \frac{1}{2} < 1$. Thus the given series is conditionally convergent.

7. $\frac{1}{2} - \frac{2}{3}\frac{1}{2^3} + \frac{3}{4}\frac{1}{3^3} - \frac{4}{5}\frac{1}{4^3} \cdots$

The series $1 + \frac{2}{3}\frac{1}{2^3} + \frac{3}{4}\frac{1}{3^3} + \frac{4}{5}\frac{1}{4^3} + \cdots$ converges, since it is term by term less than or equal to the p series with $p = 3$. Thus the given series is absolutely convergent.

8. $\frac{2}{3} - \frac{3}{4}\frac{1}{2} + \frac{4}{5}\frac{1}{3} - \frac{5}{6}\frac{1}{4} \cdots$

The series $\frac{2}{3} + \frac{3}{4}\frac{1}{2} + \frac{4}{5}\frac{1}{3} + \frac{5}{6}\frac{1}{4} + \cdots$ is divergent, being term by term greater than one-half the harmonic series. Thus the given series is conditionally convergent.

9. $2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} \cdots$

The series $2 + \frac{2^3}{3!} + \frac{2^5}{5!} + \frac{2^7}{7!} + \cdots + \frac{2^{2n-1}}{(2n-1)!} + \cdots$ is convergent (by the ratio test), and the given series is absolutely convergent.

10. $\frac{1}{2} - \frac{4}{2^3+1} + \frac{9}{3^3+1} - \frac{16}{4^3+1} \cdots$

The series $\frac{1}{2} + \frac{4}{2^3+1} + \frac{9}{3^3+1} + \frac{16}{4^3+1} + \cdots + \frac{n^2}{n^3+1} + \cdots$ is divergent (by the integral test), and the given series is conditionally convergent.

11. $\frac{1}{2} - \frac{2}{2^3+1} + \frac{3}{3^3+1} - \frac{4}{4^3+1} \cdots$

The series $\frac{1}{2} + \frac{2}{2^3+1} + \frac{3}{3^3+1} + \frac{4}{4^3+1} + \cdots + \frac{n}{n^3+1} + \cdots$ is convergent, being term by term less than the p series for $p = 2$. Thus the given series is absolutely convergent.

12. $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} \cdots$

The series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \frac{1}{4\cdot 2^4} + \cdots$ is convergent, being term by term less than or equal to the convergent geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$. Thus the given series is absolutely convergent.

Supplementary Problems

13. Examine each of the following alternating series for convergence or divergence.

(*a*) $\sum \frac{(-1)^{n-1}}{n!}$ (*b*) $\sum \frac{(-1)^{n-1}}{\ln n}$ (*c*) $\sum (-1)^{n-1} \frac{n+1}{n}$

(*d*) $\sum (-1)^{n-1} \frac{\ln n}{3n+2}$ (*e*) $\sum \frac{(-1)^{n-1}}{2n-1}$ (*f*) $\sum (-1)^{n-1} \frac{1}{\sqrt[n]{3}}$

Ans. (*a*), (*b*), (*d*), (*e*) convergent

14. Examine each of the following for conditional or absolute convergence.

(*a*) $\sum \dfrac{(-1)^{n+1}}{(2n-1)^3}$ (*b*) $\sum \dfrac{(-1)^{n-1}}{\sqrt{n(n+1)}}$ (*c*) $\sum \dfrac{(-1)^{n-1}}{(n+1)^2}$ (*d*) $\sum \dfrac{(-1)^{n-1}}{n^2+2}$

(*e*) $\sum \dfrac{(-1)^{n-1}}{3n-1}$ (*f*) $\sum \dfrac{(-1)^{n-1}}{(n!)^3}$ (*g*) $\sum (-1)^{n-1} \dfrac{n}{n^2+1}$ (*h*) $\sum (-1)^{n-1} \dfrac{n^2}{n^4+2}$

Ans. (*a*), (*c*), (*d*), (*f*), (*h*) absolutely convergent, the others conditionally convergent.

Chapter 56

Computations with Series

OPERATIONS ON SERIES. Let

$$\sum s_n = s_1 + s_2 + s_3 + \cdots + s_n + \cdots \tag{56.1}$$

be a given series, and let $\Sigma\, t_n$ be obtained from it by the insertion of parentheses. For example, one possibility is

$$\sum t_n = (s_1 + s_2) + (s_3 + s_4 + s_5) + (s_6 + s_7) + (s_8 + s_9 + s_{10} + s_{11}) + \cdots$$

Theorem 56.1: Any series obtained from a convergent series by the insertion of parentheses converges to the same sum as the original series.

Theorem 56.2: A series obtained from a divergent positive series by the insertion of parentheses diverges, but one obtained from a divergent series with mixed terms may or may not diverge.

(See Problem 1.)

Now let $\Sigma\, u_n$ be obtained from (*56.1*) by a reordering of the terms, for example, as

$$\sum u_n = s_1 + s_3 + s_2 + s_4 + s_6 + s_5 + \cdots$$

Theorem 56.3: Any series obtained from an absolutely convergent series by a reordering of the terms converges absolutely to the same sum as the original series.

Theorem 56.4: The terms of a conditionally convergent series can be rearranged to give either a divergent series or a convergent series whose sum is a preassigned number.

EXAMPLE 1: The series $\Sigma\,(-1)^{n-1}\left(\dfrac{2n+1}{n}\right)$ diverges. (Why?) When grouped as

$$\left(3-\frac{5}{2}\right)+\left(\frac{7}{3}-\frac{9}{4}\right)+\left(\frac{11}{5}-\frac{13}{6}\right)+\cdots+\left(\frac{4m-1}{2m-1}-\frac{4m+1}{2m}\right)+\cdots$$

the series converges, since the general term $\left(\dfrac{4m-1}{2m-1}-\dfrac{4m+1}{2m}\right)=\dfrac{1}{4m^2-2m}<\dfrac{1}{m^2}$.

EXAMPLE 2: The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots+\frac{1}{2n-1}-\frac{1}{2n}+\cdots$ is convergent, and it may be grouped as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n}\right)+\cdots$ to yield the convergent series $\frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\cdots=A$. When it is arranged in the pattern $+--+---\cdots$, we have $\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{4n-2}-\frac{1}{4n}\right)\cdots$ or $\frac{1}{4}+\frac{1}{24}+\frac{1}{60}+\cdots=\frac{1}{32}A$.

ADDITION, SUBTRACTION, AND MULTIPLICATION. If $\Sigma\, s_n$ and $\Sigma\, t_n$ are any two series, their *sum series* $\Sigma\, u_n$, their *difference series* $\Sigma\, v_n$, and their *product series* $\Sigma\, w_n$ are defined as

$$\sum u_n = \sum (s_n + t_n)$$

$$\sum v_n = \sum (s_n - t_n)$$

$$\sum w_n = s_1t_1 + (s_1t_2 + s_2t_1) + (s_1t_3 + s_2t_2 + s_3t_1) + \cdots$$

Theorem 56.5: If $\sum s_n$ converges to S and $\Sigma\, t_n$ converges to T, then $\Sigma\,(s_n + t_n)$ converges to $S+T$ and $\Sigma\,(s_n - t_n)$ converges to $S-T$. If $\Sigma\, s_n$ and $\Sigma\, t_n$ are both absolutely convergent, so also are $\Sigma\,(s_n \pm t_n)$.

(See Problems 2 and 3.)

Theorem 56.6: If Σs_n and Σt_n converge, their product series Σw_n may or may not converge. If Σs_n and Σt_n converge and at least one of them is absolutely convergent, then Σw_n converges to ST. If Σs_n and Σt_n are absolutely convergent, so also is Σw_n.

COMPUTATIONS WITH SERIES. The sum of a convergent series can be obtained readily provided the nth partial sum can be expressed as a function of n; for example, any convergent geometric series. On the other hand, any partial sum of a convergent series may be taken as an approximation of the sum of the series. If the approximation S_n of S is to be useful, information concerning the possible size of $|S_n - S|$ must be known.

For a convergent series Σs_n with sum S, we write

$$S = S_n + R_n$$

where R_n, called the *remainder after n terms*, is the error introduced by using S_n, the nth partial sum, instead of the true sum S. The theorems below give approximations of this error in the form $R_n < \alpha$ for positive series and $|R_n| \leq \alpha$ for series with mixed terms.

For a convergent alternating series $s_1 - s_2 + s_3 - s_4 + \cdots$,

$$R_{2m} = s_{2m+1} - s_{2m+2} + s_{2m+3} - s_{2m+4} + \cdots < s_{2m+1}$$

and

$$R_{2m+1} = -s_{2m+2} + s_{2m+3} - s_{2m+4} + s_{2m+5} - \cdots > -s_{2m+2}$$

by Problem 1 of Chapter 55. Thus, we have:

Theorem 56.7: For a convergent alternating series, $|R_n| < s_{n+1}$; moreover, R_n is positive when n is even, and R_n is negative when n is odd.

(See Problem 4.)

Theorem 56.8: For the convergent geometric series Σar^{n-1}, $|R_n| = \left|\dfrac{ar^n}{1-r}\right|$.

Theorem 56.9: If the positive series Σs_n converges by the integral test, then

$$R_n < \int_n^{+\infty} f(x)\, dx$$

(See Problems 5 to 7.)

Theorem 56.10: If Σc_n is a known convergent positive series, and if for the positive series Σs_n, $s_n \leq c_n$ for every value of $n > n_1$, then

$$R_n \leq \sum_{n+1}^{+\infty} c_j \qquad \text{for } n > n_1$$

(See Problems 8 to 10.)

Solved Problems

1. Let $\Sigma s_n = s_1 + s_2 + s_3 + \cdots + s_n + \cdots$ be a given positive series, and let $\Sigma t_n = (s_1 + s_2) + s_3 + (s_4 + s_5) + s_6 + \cdots$ be obtained from it by the insertion of parentheses according to the pattern 2, 1, 2, 1, 2, 1, ... Discuss the convergence or divergence of Σt_n.

For the partial sums of Σt_n, we have $T_1 = S_2$, $T_2 = S_3$, $T_3 = S_5$, $T_4 = S_6, \ldots$. If Σs_n converges to S so also does Σt_n, since $\lim\limits_{n\to+\infty} T_n = \lim\limits_{n\to+\infty} S_n$. If Σs_n diverges, $\{S_n\}$ is unbounded and so also is $\{T_n\}$; hence Σt_n diverges.

2. Show that $\dfrac{3+1}{3\cdot 1}+\dfrac{3^2+2^3}{3^2\cdot 2^3}+\dfrac{3^3+3^3}{3^3\cdot 3^3}+\cdots+\dfrac{3^n+n^3}{3^n\cdot n^3}+\cdots$ converges.

Since $\dfrac{3^n+n^3}{3^n\cdot n^3}=\dfrac{1}{n^3}+\dfrac{1}{3^n}$, the given series is the sum of the two series $\sum \dfrac{1}{n^3}$ and $\sum \dfrac{1}{3^n}$. Each is convergent; hence by Theorem 56.5 the given series converges.

3. Show that the series $\dfrac{3^n+n}{n\cdot 3^n}$ diverges.

Suppose $\sum \dfrac{3^n+n}{n\cdot 3^n}=\sum\left(\dfrac{1}{n}+\dfrac{1}{3^n}\right)$ converges. Then, since $\sum \dfrac{1}{3^n}$ converges, so also (by Theorem 56.5) does $\sum \dfrac{1}{n}$. But this is false; hence the given series diverges.

4. (*a*) Estimate the error when $\Sigma\, s_n = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}\cdots$ is approximated by its first 10 terms.
(*b*) How many terms must be used to compute the value of the series with allowable error 0.05?

(*a*) This is a convergent alternating series. The error $R_{10}<s_{11}=1/11^2=0.0083$.

(*b*) Since $|R_n|<s_{n+1}$, set $s_{n+1}=\dfrac{1}{(n+1)^2}=0.05$. Then $(n+1)^2=20$ and $n=3.5$. Hence four terms are required.

5. Establish $R_n<\displaystyle\int_n^{+\infty} f(x)\,dx$ as given in Theorem 56.9.

In Fig. 54-1, let the approximation (by the smaller rectangles) of the area under the curve be extended to the right of $x=n$. Then

$$R_n=s_{n+1}+s_{n+2}+s_{n+3}+\cdots<\int_n^{+\infty} f(x)\,dx$$

6. Estimate the error when $\sum \dfrac{1}{4n^2}$ is approximated by its first 10 terms.

This series converges by the integral test (Problem 3 of Chapter 54). Then

$$R_{10}<\frac{1}{4}\int_{10}^{+\infty}\frac{dx}{x^2}=\frac{1}{4}\lim_{u\to+\infty}\int_{10}^{u}\frac{dx}{x^2}=\frac{1}{4}\lim_{u\to+\infty}\left(-\frac{1}{u}+\frac{1}{10}\right)=\frac{1}{40}=0.025$$

7. Estimate the number of terms necessary to compute $\sum \dfrac{1}{n^5+1}$ with allowable error 0.00001.

This series converges by comparison with $\sum \dfrac{1}{n^5}$ which, in turn, converges by the integral test. Then $R_n<\displaystyle\int_n^{+\infty}\frac{dx}{x^5}=\frac{1}{4n^4}$. Setting $\dfrac{1}{4n^4}=0.00001$, we find $n^4=25{,}000$ and $n=12.6$. Thus 13 terms are necessary.

8. Estimate the error when $\sum \dfrac{1}{n!}$ is approximated by its first 12 terms.

This series was found to converge (in Problem 8 of Chapter 54) by comparison with the geometric series $\sum \dfrac{1}{2^{n-1}}$. Thus the error R_{12} for the given series is less than the error R'_{12} for the geometric series; that is, $R_{12}<R'_{12}=\dfrac{(1/2)^{12}}{1-1/2}=\dfrac{1}{2^{11}}=0.0005$.

We can do better! For $n>6$, $\dfrac{1}{n!}<\dfrac{1}{4^{n-1}}$; hence, $R_{12}<\dfrac{(1/4)^{12}}{1-1/4}=\dfrac{1}{3(4^{11})}=0.000\,000\,08$.

9. Estimate the error when $\Sigma s_n = \frac{2}{3} + \frac{1}{2}(\frac{2}{3})^2 + \frac{1}{3}(\frac{2}{3})^3 + \frac{1}{4}(\frac{2}{3})^4 + \cdots$ is approximated by its first 10 terms.

The series converges by the ratio test, since $\frac{s_{n+1}}{s_n} = \frac{2}{3}\frac{n}{n+1}$ and $r = \lim_{n\to+\infty} \frac{s_{n+1}}{s_n} = \frac{2}{3}$. Now $\frac{s_{n+1}}{s_n} < \frac{2}{3}$ for every value of n, so that the given series is term by term less than or equal to the geometric series $\Sigma s_1 r^{n-1}$. Hence $R_{10} < \left(\frac{2}{3}\right)^{11} + \left(\frac{2}{3}\right)^{12} + \left(\frac{2}{3}\right)^{13} + \cdots = \frac{(2/3)^{11}}{1-2/3} = \frac{2^{11}}{3^{10}} = 0.04$.

A better approximation may be obtained by noting that after the tenth term the given series is term by term less than $\sum s_{11}\left(\frac{2}{3}\right)^{n-1} = \sum \frac{1}{11}\left(\frac{2}{3}\right)^{11}\left(\frac{2}{3}\right)^{n-1} = \frac{2^{11}}{11\cdot 3^{10}} = 0.004$.

10. Estimate the error when $\sum s_n = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$ is approximted by its first 10 terms.

The series converges by the ratio test, since $\frac{s_{n+1}}{s_n} = \frac{1}{3}\frac{n+1}{n}$ and $r = \frac{1}{3}$. Here $\frac{s_{n+1}}{s_n} \geq \frac{1}{3}$ for every value of n, and we cannot use the geometric series $\Sigma\,(\frac{1}{3})^n$ as comparison series. However, $\left\{\frac{s_{n+1}}{s_n}\right\}$ is a nonincreasing sequence, and $\frac{s_{12}}{s_{11}} = \frac{4}{11}$; hence after the first 10 terms the given series is term by term less than or equal to the geometric series $\sum s_{11}\left(\frac{4}{11}\right)^{n-1} = \frac{11}{3^{11}}\left(\frac{4}{11}\right)^{n-1}$. Then $R_{10} < \sum \frac{11}{3^{11}}\left(\frac{4}{11}\right)^{n-1} = \frac{121}{7\cdot 3^{11}} = 0.000\,097\,58 < 0.0001$.

Supplementary Problems

11. Rearrange the terms of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots$ to produce a convergent series whose sum is (*a*) 1, (*b*) -2.
(*Hint*: In (*a*), write the first n_1 positive terms until their sum first exceeds 1, then follow with the first n_2 negative terms until the sum first falls below 1, and repeat.)

12. Can the sum of two divergent series converge? Give an example.

Ans. yes; a trivial example is $\sum \frac{1}{n} + \sum \frac{-1}{n}$

13. (*a*) Estimate the error when the series $\sum \frac{(-1)^{n-1}}{2n-1}$ is approximated by its first 50 terms.
(*b*) Estimate the number of terms necessary to compute the sum if the allowable error is 0.000 005.

Ans. (*a*) 0.01; (*b*) 100,000

14. (*a*) Estimate the error when $\sum \frac{(-1)^{n-1}}{n^4}$ is approximated by its first eight terms.
(*b*) Estimate the number of terms necessary to compute the sum if the allowable error is 0.00005.

Ans. (*a*) 0.0002; (*b*) 11

15. (*a*) Estimate the error when the geometric series $\sum \frac{3}{2^n}$ is approximated by its first six terms.
(*b*) How many terms are necessary to compute the sum if the allowable error is 0.00005?

Ans. (*a*) 0.05; (*b*) 16

16. Prove: If the positive series Σs_n converges by comparison with the geometric series Σr^n, for $0 < r < 1$, then $R_n < \frac{r^{n+1}}{1-r}$.

17. Estimate the error when (*a*) $\sum \frac{1}{3^n+1}\left(<\sum \frac{1}{3^n}\right)$ is approximated by its first six terms; (*b*) $\sum \frac{1}{3+4^n}\left(<\sum \frac{1}{4^n}\right)$ is approximated by its first six terms.

Ans. (*a*) 0.0007; (*b*) 0.00009

18. The series (*a*) $\sum \frac{n+1}{n\cdot 3^n}$ and (*b*) $\sum \frac{n}{(n+1)3^n}$ are convergent by the ratio test. Estimate the error when each is approximated by its first eight terms. *Ans.* (*a*) 0.00009; (*b*) 0.00007

19. For the convergent p series, show that $R_n < \frac{1}{(p-1)n^{p-1}}$. (*Hint*: See Problem 7.)

20. The series (*a*) $\sum \frac{1}{n^3+2}$ and (*b*) $\sum \frac{n-1}{n^5}$ are convergent by comparison with appropriate p series. Estimate the error when each is approximated by its first six terms, and find the number of terms needed for the sum if the allowable error is 0.005. *Ans.* (*a*) 0.014, 10 terms; (*b*) 0.002, 5 terms

Chapter 57

Power Series

AN INFINITE SERIES of the form

$$\sum c_i x^i = \sum_{i=0}^{+\infty} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots \tag{57.1}$$

where the c's are constants, is called a *power series in x*. Similarly, an infinite series of the form

$$\sum c_i (x-a)^i = \sum_{i=0}^{+\infty} c_i (x-a)^i = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \tag{57.2}$$

is called a *power series in* $(x - a)$.

For any given value of x, both (*57.1*) and (*57.2*) become infinite series of constant terms and (see Chapters 54 and 55) either converge or diverge.

INTERVAL OF CONVERGENCE. The totality of values of x for which a power series converges is called its *interval of convergence*. Clearly, (*57.1*) converges for $x = 0$ and (*57.2*) converges for $x = a$. If there are other values of x for which a power series (*57.1*) or (*57.2*) converges, then it converges either for all values of x or for all values of x on some finite interval (closed, open, or half-open) having as midpoint $x = 0$ for (*57.1*) or $x = a$ for (*57.2*).

The interval of convergence will be found here by using the ratio test for absolute convergence supplemented by other tests of Chapters 54 and 55 at the endpoints. (See Problems 1 to 9.)

CONVERGENCE AND UNIFORM CONVERGENCE. The discussion and theorems given below involve series of the type of (*57.1*) but apply equally after only minor changes to series of the type of (*57.2*).

Consider the power series (*57.1*). Denote by

$$S_n(x) = \sum_{j=0}^{n-1} c_j x^j = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$$

the nth *partial sum* and by

$$R_n(x) = \sum_{k=n}^{+\infty} c_k x^k = c_n x^n + c_{n+1} x^{n+1} + c_{n+2} x^{n+2} + \cdots$$

the *remainder after n terms*. Then

$$\sum c_i x^i = S_n(x) + R_n(x) \tag{57.3}$$

If for $x = x_0$, $\Sigma\, c_i x^i$ converges to $S(x_0)$, a finite number, then $\lim_{n\to+\infty} S_n(x_0) = S(x_0)$. Since $|S(x_0) - S_n(x_0)| = |R_n(x_0)|$, $\lim_{n\to+\infty} |S(x_0) - S_n(x_0)| = \lim_{n\to+\infty} |R_n(x_0)| = 0$. Thus, $\Sigma\, c_i x^i$ converges for $x = x_0$ if for any positive ϵ, however small, there exists a positive integer m such that whenever $n > m$ then $|R_n(x_0)| < \epsilon$.

Note that here m depends not only upon ϵ (see Problem 12 of Chapter 53) but also upon the choice x_0 of x. (See Problem 10.)

In Problem 11, we prove the first of our theorems:

Theorem 57.1: If $\Sigma\, c_i x^i$ converges for $x = x_1$, and if $|x_2| < |x_1|$, then the series converges absolutely for $x = x_2$.

Suppose now that (*57.1*) converges absolutely, that is, $\Sigma\, |c_i x^i|$ converges, for all values of x such that $|x| < P$. Choose a value of x, either $x = p$ or $x = -p$, so that $|x| = p < P$. Since (*57.1*) converges for $|x| = p$, it follows that for any $\epsilon > 0$, however small, there exists a positive integer m such that whenever $n > m$, then $|R_n(p)| = \sum_{k=n}^{+\infty} |c_k p^k| < \epsilon$. Now let x vary over the interval $|x| \le p$. Every term of $|R_n(x)| = \sum_{k=n}^{+\infty} |c_k x^k|$ has its maximum value at $|x| = p$; hence $|R_n(x)|$ has its maximum value on the interval $|x| \le p$ when $|x| = p$.

Let ϵ be chosen and m be found when $|x| = p$. Then for this ϵ and m, $|R_n(x)| < \epsilon$ for *all* x such that $|x| \le p$; that is, m depends on ϵ and p but not on the choice x_0 of x on the interval $|x| \le p$ as in ordinary convergence. We say that (*57.1*) is *uniformly convergent* on the interval $|x| \le p$. We have proved

Theorem 57.2: If $\Sigma\, c_i x^i$ converges absolutely for $|x| < P$, then it converges uniformly for $|x| \le p < P$.

As an example, the series $\Sigma\, (-1)^i x^i$ is convergent for $|x| < 1$. By Theorem 57.1 it is absolutely convergent for $|x| \le 0.99$, and by Theorem 57.2 it is uniformly convergent for $|x| \le 0.9$.

Theorem 57.3: A power series *represents* a continuous function $f(x)$ *within* the interval of convergence of the series.

(For a proof, see Problem 12.)

Theorem 57.4: If $\Sigma\, c_i x^i$ converges to the function $f(x)$ on an interval I, and if a and b are *within* the interval, then

$$\int_a^b f(x)\,dx = \sum_{i=0}^{+\infty} \int_a^b c_i x^i\,dx$$
$$= \int_a^b c_0\,dx + \int_a^b c_1 x\,dx + \int_a^b c_2 x^2\,dx + \cdots + \int_a^b c_{n-1} x^{n-1}\,dx + \cdots$$

(For a proof, see Problem 13.)

Theorem 57.5: If $\Sigma\, c_i x^i$ converges to $f(x)$ on an interval I, then the indefinite integral $\sum_{i=0}^{+\infty} \int_0^x c_i x^i\,dx$ converges to $g(x) = \int_0^x f(x)\,dx$ for all x *within* the interval I.

Theorem 57.6: If $\Sigma\, c_i x^i$ converges to the function $f(x)$ on the interval I, then the term-by-term derivative of the series, $\sum \frac{d}{dx}(c_i x^i)$ converges to $f'(x)$ for all x *within* the interval I.

Theorem 57.7: The representation of a function $f(x)$ in powers of x is unique.

Solved Problems

1. Find the interval of convergence of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n \cdots$.

The ratio test yields

$$\lim_{n\to+\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to+\infty}\left|\frac{x^{n+1}}{n+1}\,\frac{n}{x^n}\right| = |x| \lim_{n\to+\infty}\frac{n}{n+1} = |x|$$

The series converges absolutely for $|x|<1$ and diverges for $|x|>1$. Individual tests *must* be made at the endpoints $x=1$ and $x=-1$:

For $x=1$, the series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$ and is conditionally convergent.
For $x=-1$, the series becomes $-(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots)$ and is divergent.
Thus the given series converges on the interval $-1<x\le 1$.

2. Find the interval of convergence of $1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\cdots+\dfrac{x^n}{n!}+\cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{s_{n+1}}{s_n}\right|=\lim_{n\to+\infty}\left|\frac{x^{n+1}}{(n+1)!}\,\frac{n!}{x^n}\right|=|x|\lim_{n\to+\infty}\frac{1}{n+1}=0$$

The given series converges for all values of x.

3. Find the interval of convergence of $\dfrac{x-2}{1}+\dfrac{(x-2)^2}{2}+\dfrac{(x-2)^3}{3}+\cdots+\dfrac{(x-2)^n}{n}+\cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x-2)^{n+1}}{n+1}\,\frac{n}{(x-2)^n}\right|=|x-2|\lim_{n\to+\infty}\frac{n}{n+1}=|x-2|$$

The series converges absolutely for $|x-2|<1$ or $1<x<3$ and diverges for $|x-2|>1$ or for $x<1$ and $x>3$.

For $x=1$ the series becomes $-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots$, and for $x=3$ it becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$. The first converges, and the second diverges. Thus the given series converges on the interval $1\le x<3$ and diverges elsewhere.

4. Find the interval of convergence of $1+\dfrac{x-3}{1^2}+\dfrac{(x-3)^2}{2^2}+\dfrac{(x-3)^3}{3^2}+\cdots+\dfrac{(x-3)^{n-1}}{(n-1)^2}+\cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x-3)^n}{n^2}\,\frac{(n-1)^2}{(x-3)^{n-1}}\right|=|x-3|\lim_{n\to+\infty}\left(\frac{n-1}{n}\right)^2=|x-3|$$

The series converges absolutely for $|x-3|<1$ or $2<x<4$ and diverges for $|x-3|>1$ or for $x<2$ and $x>4$.

For $x=2$ the series becomes $1-1+\frac{1}{4}-\frac{1}{9}+\cdots$, and for $x=4$ it becomes $1+1+\frac{1}{4}+\frac{1}{9}+\cdots$. Since both are absolutely convergent, the given series converges absolutely on the interval $2\le x\le 4$ and diverges elsewhere. Note that the first term of the series is not given by the general term with $n=0$.

5. Find the interval of convergence of $\dfrac{x+1}{\sqrt{1}}+\dfrac{(x+1)^2}{\sqrt{2}}+\dfrac{(x+1)^3}{\sqrt{3}}+\cdots+\dfrac{(x+1)^n}{\sqrt{n}}+\cdots$.

Here $$\lim_{n\to+\infty}\left|\frac{(x+1)^{n+1}}{\sqrt{n+1}}\,\frac{\sqrt{n}}{(x+1)^n}\right|=|x+1|\lim_{n\to+\infty}\sqrt{\frac{n}{n+1}}=|x+1|$$

The series converges absolutely for $|x+1|<1$ or $-2<x<0$ and diverges for $x<-2$ and $x>0$.

For $x=-2$ the series becomes $-1+\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}\cdots$, and for $x=0$ it becomes $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\cdots$. The first is convergent, and the second is divergent (why?). Thus, the given series converges on the interval $-2\le x<0$ and diverges elsewhere.

6. Find the interval of convergence of $1+\dfrac{m}{1}x+\dfrac{m(m-1)}{1\cdot 2}x^2+\dfrac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3+\cdots$.

This is the binomial series. For positive integer values of m, the series is finite; for all other values of m, it is an infinite series. We have

$$\lim_{n\to+\infty}\left|\frac{m(m-1)(m-2)\cdots(m-n+1)x^n}{n!}\;\frac{(n-1)!}{m(m-1)(m-2)\cdots(m-n+2)x^{n-1}}\right|$$
$$=|x|\lim_{n\to+\infty}\left|\frac{m-n+1}{n}\right|=|x|$$

The infinite series converges absolutely for $|x|<1$ and diverges for $|x|>1$.

At the endpoints $x=\pm 1$, the series converges when $m \geq 0$ and diverges when $m \leq -1$. When $-1<m<0$, the series converges when $x=1$ and diverges when $x=-1$. To establish these facts, tests more delicate than those of Chapter 54 are needed.

7. Find the interval of convergence of $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots+(-1)^{n-1}\dfrac{x^{2n-1}}{2n-1}+\cdots$.

Here
$$\lim_{n\to+\infty}\left|\frac{x^{2n+1}}{2n+1}\,\frac{2n-1}{x^{2n-1}}\right| = x^2\lim_{n\to+\infty}\frac{2n-1}{2n+1}=x^2$$

The series is absolutely convergent on the interval $x^2<1$ or $-1<x<1$.

For $x=-1$ the series becomes $-1+\frac{1}{3}-\frac{1}{5}+\frac{1}{7}\cdots$, and for $x=1$ it becomes $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots$. Both series converge; thus the given series converges for $-1\leq x\leq 1$ and diverges elsewhere.

8. Find the interval of convergence of $(x-1)+2!(x-1)^2+3!(x-1)^3+\cdots+n!(x-1)^n+\cdots$.

Here
$$\lim_{n\to+\infty}\left|\frac{(n+1)!(x-1)^{n+1}}{n!(x-1)^n}\right| = |x-1|\lim_{n\to+\infty}(n+1)=\infty$$

The series converges for $x=1$ only.

9. Find the interval of convergence of $\dfrac{1}{2x}+\dfrac{2}{4x^2}+\dfrac{3}{8x^3}+\cdots+\dfrac{n}{2^nx^n}+\cdots$. This is a power series in $1/x$.

Here
$$\lim_{x\to+\infty}\left|\frac{n+1}{2^{n+1}x^{n+1}}\,\frac{2^nx^n}{n}\right| = \frac{1}{2|x|}\lim_{n\to+\infty}\frac{n+1}{n}=\frac{1}{2|x|}$$

The series converges absolutely for $\dfrac{1}{2|x|}<1$ or $|x|>\frac{1}{2}$.

For $x=\frac{1}{2}$ the series becomes $1+2+3+4+\cdots$ and for $x=-\frac{1}{2}$ the series becomes $-1+2-3+4\cdots$. Both these series diverge. Thus the given series converges on the intervals $x<-\frac{1}{2}$ and $x>\frac{1}{2}$ and diverges on the interval $-\frac{1}{2}\leq x\leq\frac{1}{2}$.

10. The series $1-x+x^2-x^3\cdots+(-1)^nx^n+\cdots$ converges for $|x|<1$. Given $\epsilon=0.000\,001$, find m when (*a*) $x=\frac{1}{2}$ and (*b*) $x=\frac{1}{4}$ so that $|R_n(x)|<\epsilon$ for $n>m$.

$R_n(x)=\sum\limits_{k=n}^{+\infty}(-1)^kx^k$, so that

$$|R_n(\tfrac{1}{2})|=\left|\sum_{k=n}^{+\infty}(-1)^k(\tfrac{1}{2})^k\right|=\tfrac{1}{3}(\tfrac{1}{2})^{n-1}\qquad\text{and}\qquad |R_n(\tfrac{1}{4})|=\left|\sum_{k=n}^{+\infty}(-1)^k(\tfrac{1}{4})^k\right|=\tfrac{1}{5}(\tfrac{1}{4})^{n-1}$$

(*a*) We seek m such that for $n>m$ then $\frac{1}{3}(\frac{1}{2})^{n-1}<0.000\,001$ or $1/2^{n-1}<0.000\,003$. Since $1/2^{18}=0.000\,004$ and $1/2^{19}=0.000\,002$, $m=19$.

(*b*) We seek m such that for $n>m$ then $\frac{1}{5}(\frac{1}{4})^{n-1}<0.000\,001$ or $1/4^{n-1}<0.000\,005$. Here, $m=9$.

11. Prove: If a power series $\Sigma\, c_ix^i$ converges for $x=x_1$ and if $|x_2|<|x_1|$, the series converges absolutely for $x=x_2$.

Since $\Sigma\, c_ix_1^i$ converges, $\lim\limits_{n\to+\infty} c_nx_1^n=0$ by Theorem 53.15; also $\{|c_ix_1^i|\}$, being convergent, is bounded, say, $0<|c_nx_1^n|<K$ for all values of n. Suppose $|x_2/x_1|=r$, for $0<r<1$; then

$$|c_nx_2^n|=|c_nx_1^n|\left|\frac{x_2^n}{x_1^n}\right|=|c_nx_1^n|\left|\frac{x_2}{x_1}\right|^n<Kr^n$$

and $\Sigma\,|c_nx_2^n|$, being term by term less than the convergent geometric series $\Sigma\, Kr^n$, is convergent. Thus $\Sigma\, c_ix_2^i$ converges and, in fact, converges absolutely.

12. Prove: A power series represents a continuous function $f(x)$ within the interval of convergence of the series.

Set $f(x) = \Sigma\, c_i x^i = S_n(x) + R_n(x)$. For *any* $x = x_0$ within the interval of convergence of $\Sigma\, c_i x^i$ there is, by Theorem 57.1, an interval I about x_0 on which the series is uniformly convergent. To prove $f(x)$ continuous at $x = x_0$, it is necessary to show that $\lim\limits_{\Delta x \to 0} |f(x_0 + \Delta x) - f(x_0)| = 0$ when $x_0 + \Delta x$ is on I; that is, it is necessary to show that for a given $\epsilon > 0$, however small, Δx may be chosen so that $x_0 + \Delta x$ is on I and $|f(x_0 + \Delta x) - f(x_0)| < \epsilon$.

Now for any Δx such that $x_0 + \Delta x$ is on the interval I,

$$\begin{aligned} |f(x_0 + \Delta x) - f(x_0)| &= |S_n(x_0 + \Delta x) + R_n(x_0 + \Delta x) - S_n(x_0) - R_n(x_0)| \\ &\le |S_n(x_0 + \Delta x) - S_n(x_0)| + |R_n(x_0 + x)| + |R_n(x_0)| \end{aligned} \tag{1}$$

Let ϵ be chosen. Since $x_0 + \Delta x$ is on the interval of convergence of the series, an integer $m > 0$ can be found so that whenever $n > m$ then $|R_n(x_0 + \Delta x)| < \epsilon/3$ and $|R_n(x_0)| < \epsilon/3$. Also, since $S_n(x)$ is a polynomial, a smaller $|\Delta x|$ can be chosen, if necessary, so that $|S_n(x_0 + \Delta x) - S_n(x_0)| < \epsilon/3$. For this new choice of Δx, $|R_n(x_0 + \Delta x)|$ remains less than $\epsilon/3$ since the series is uniformly convergent on I and $|R_n(x_0)|$ is unchanged. Hence, by (*1*),

$$|f(x_0 + \Delta x) - f(x_0)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

Thus $f(x)$ is continuous for all x within the interval of convergence of the series.

13. Prove: If $\Sigma\, c_i x^i$ converges to the function $f(x)$ on an interval, and if $x = a$ and $x = b$ are within the interval, then

$$\int_a^b f(x)\,dx = \int_a^b c_0\,dx + \int_a^b c_1 x\,dx + \int_a^b c_2 x^2\,dx + \cdots + \int_a^b c_{n-1} x^{n-1}\,dx + \cdots$$

Suppose $b > a$ and write $f(x) = \Sigma\, c_i x^i = S_n(x) + R_n(x)$. Then

$$\int_a^b f(x)\,dx = \int_a^b S_n(x)\,dx + \int_a^b R_n(x)\,dx$$

and

$$\left| \int_a^b f(x)\,dx - \int_a^b S_n(x)\,dx \right| = \left| \int_a^b R_n(x)\,dx \right|$$

Since $\Sigma\, c_i x^i$ is convergent on an interval, say $|x| < P$, the series is uniformly convergent on an interval $|x| \le p < P$ which includes both $x = a$ and $x = b$. Then for any $\epsilon > 0$, however small, n can be chosen sufficiently large that $|R_n(x)| < \dfrac{\epsilon}{b - a}$ for all $|x| \le p$. Thus,

$$\left| \int_a^b f(x)\,dx - \int_a^b S_n(x)\,dx \right| < \int_a^b \frac{\epsilon}{b-a}\,dx = \frac{\epsilon}{b-a}(b - a) = \epsilon$$

So

$$\lim_{n \to +\infty} \left| \int_a^b f(x)\,dx - \int_a^b S_n(x)\,dx \right| = 0 \qquad \text{and} \qquad \int_a^b f(x)\,dx = \sum \int_a^b c_i x^i\,dx$$

as was to be proved.

Supplementary Problems

14. Find the interval of convergence of each of the following series.

(*a*) $x + 2x^2 + 3x^3 + 4x^4 + \cdots$

(*b*) $\dfrac{x}{1 \cdot 2} + \dfrac{x^2}{2 \cdot 3} + \dfrac{x^3}{3 \cdot 4} + \dfrac{x^4}{4 \cdot 5} + \cdots$

(*c*) $x - \dfrac{x^2}{2^2} + \dfrac{x^3}{3^3} - \dfrac{x^4}{4^4} + \cdots$

(*d*) $\dfrac{x}{5} - \dfrac{x^2}{2 \cdot 5^2} + \dfrac{x^3}{3 \cdot 5^3} - \dfrac{x^4}{4 \cdot 5^4} + \cdots$

(*e*) $\frac{1}{1\cdot 2\cdot 3}+\frac{x^2}{2\cdot 3\cdot 4}+\frac{x^4}{3\cdot 4\cdot 5}+\frac{x^6}{4\cdot 5\cdot 6}+\cdots$ (*f*) $\frac{x^2}{(\ln 2)^2}+\frac{x^3}{(\ln 3)^3}+\frac{x^4}{(\ln 4)^4}+\frac{x^5}{(\ln 5)^5}+\cdots$

(*g*) The series obtained by differentiating (*a*) term by term
(*h*) The series obtained by differentiating (*b*) term by term

(*i*) $x+\frac{x^2}{1+2^3}+\frac{x^3}{1+3^3}+\frac{x^4}{1+4^3}+\cdots$

(*j*) The series obtained by differentiating (*i*) term by term
(*k*) The series obtained by differentiating (*j*) term by term
(*l*) The series obtained by integrating (*a*) term by term
(*m*) The series obtained by integrating (*c*) term by term

(*n*) $(x-2)+\frac{(x-2)^2}{4}+\frac{(x-2)^3}{9}+\frac{(x-2)^4}{16}+\cdots$

(*o*) $\frac{x-3}{1\cdot 3}+\frac{(x-3)^2}{2\cdot 3^2}+\frac{(x-3)^3}{3\cdot 3^3}+\frac{(x-3)^4}{4\cdot 3^4}+\cdots$

(*p*) $1-\frac{3x-2}{5}+\frac{(3x-2)^2}{5^2}+\frac{(3x-2)^3}{5^3}+\cdots$

(*q*) The series obtained by differentiating (*a*) term by term
(*r*) The series obtained by integrating (*n*) term by term

(*s*) $1+\frac{x}{1-x}+\left(\frac{x}{1-x}\right)^2+\left(\frac{x}{1-x}\right)^3+\cdots$

(*t*) $1-\frac{2}{x}+\frac{3}{x^2}-\frac{4}{x^3}+\cdots$

(*u*) $\frac{1}{2}+\frac{x^2+6x+7}{2^2}+\frac{(x^2+6x+7)^2}{2^3}+\frac{(x^2+6x+7)^3}{2^4}+\cdots$

Ans. (*a*) $-1<x<1$; (*b*) $-1\le x\le 1$; (*c*) all values of x; (*d*) $-5<x\le 5$; (*e*) $-1\le x\le 1$; (*f*) all values of x; (*g*) $-1<x<1$; (*h*) $-1\le x<1$; (*i*) $-1\le x\le 1$; (*j*) $-1\le x\le 1$; (*k*) $-1\le x<1$; (*l*) $-1<x<1$; (*m*) all values of x; (*n*) $1\le x\le 3$; (*o*) $0\le x<6$; (*p*) $-1<x<\frac{7}{3}$; (*q*) $1\le x<3$; (*r*) $1\le x\le 3$; (*s*) $x<\frac{1}{2}$; (*t*) $x<-1$, $x>1$; (*u*) $-5<x<-3$, $-3<x<-1$

15. Prove: A power series can be differentiated term by term within its interval of convergence. (*Hint*: $f(x)=\sum_{i=0}^{+\infty} c_i x^i$ and $\sum_{i=0}^{+\infty}\frac{d}{dx}(c_i x^i)=\sum_{j=1}^{+\infty} jc_j x^{j-1}$ converge for $|x|<\lim_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right|$. Use Theorems 57.1, 57.2, and 57.5 to show $\int_0^x f'(x)\,dx=f(x)$.)

16. Prove: The representation of a function $f(x)$ in powers of x is unique. (*Hint*: Let $f(x)=\Sigma\, s_n x^n$ and $f(x)=\Sigma\, t_n x^n$ on $|x|<a\neq 0$. Put $x=0$ in $\Sigma\,(s_n-t_n)x^n=0$, $\frac{d}{dx}\Sigma\,(s_n-t_n)x^n=0$, $\frac{d^2}{dx^2}\Sigma\,(s_n-t_n)x^n=0,\ldots$ to obtain $s_j=t_j$, $j=0, 1, 2, 3, \ldots$.)

Chapter 58

Series Expansion of Functions

POWER SERIES in x may be generated in various ways; for example, imagining the division continued indefinitely, we find that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots \qquad (58.1)$$

(Note that for, say, $x = 5$ this is a perfectly absurd statement.) In Example 1 below, it is shown that the series (*58.1*) represents $\frac{1}{1-x}$ only on the interval $|x| < 1$; that is,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots \qquad -1 < x < 1$$

Other methods for generating power series are illustrated below and in Problem 1.

A GENERAL METHOD for expanding a function in a powers series in x and in $(x - a)$ is given below. Note the requirement that the function and its derivatives of *all* orders must exist at $x = 0$ or at $x = a$. Thus $1/x$, $\ln x$, and $\cot x$ *cannot* be expanded in powers of x.

Maclaurin's series: Assuming that a given function can be represented by a power series in x, that series is necessarily of the form of *Maclaurin's series*:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \cdots \qquad (58.2)$$

Taylor's series: Assuming that a given function can be represented by a power series in $(x - a)$, that series is necessarily of the form of *Taylor's series*:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \cdots \qquad (58.3)$$

(See Problem 2.4.)

The question of the interval on which $f(x)$ is represented by its Maclaurin's or Taylor's series will be considered in the next chapter. For the functions of this book, the interval on which a series represents the function coincides with the interval of convergence of the series (See Problems 3 to 9.)

Another and very useful form of Taylor's series

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots \qquad (58.4)$$

is obtained by replacing x by $a + h$ in (*58.3*).

EXAMPLE 1: The power series $1 + x + x^2 + x^3 + \cdots + x^{n-1} + \cdots$ is an infinite geometric series with $a = 1$ and $r = x$. For $|r| = |x| < 1$, the series converges to $\frac{a}{1-r} = \frac{1}{1-x}$; for $|r| = |x| \geq 1$, the series diverges.

By repeated differentiation of the series of Example 1, we obtain other power series,

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \qquad (58.5)$$

$$2 + 6x + 12x^2 + 20x^3 + \cdots + n(n+1)x^{n-1} + \cdots \qquad (58.6)$$

By Theorem 57.6, the series (*58.5*) represents the function $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$ in the interval $|x| < 1$, and (*58.6*) represents the function $\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = -\frac{2}{(1-x)^3}$ in the same interval.

By repeated integration between the limits 0 and x of the series of Example 1, we obtain

$$x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots + \frac{1}{n}x^n + \cdots \tag{58.7}$$

$$\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5 + \cdots + \frac{1}{n(n+1)}x^{n+1} + \cdots \tag{58.8}$$

By Theorem 57.5, the series (*58.7*) represents the function $\int_0^x \frac{1}{1-x}\,dx = -\ln(1-x)$ in the interval $|x| < 1$. The series (*58.7*) also converges for the endpoint $x = -1$. In such a case, and where the function that is represented inside the interval is continuous at an endpoint, the function is equal to the series at the endpoint also. (The proof of this fact is beyond the scope of this book.) Hence, $-\ln 2 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\cdots$, and, therefore, $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots$.

Similarly, the series (*58.8*) represents the function $\int_0^x -\ln(1-x)\,dx = x + (1-x)\ln(1-x)$ in the interval $-1 \le x < 1$.

Solved Problems

1. Find the power series $y = \Sigma\, c_n x^n$ satisfying the conditions $y = 2$ when $x = 0$, $y' = 1$ when $x = 0$, and $y'' + 2y' = 0$.

Consider

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots \tag{1}$$
$$y' = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \cdots \tag{2}$$
$$y'' = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots \tag{3}$$

From (*1*) with $x = 0$ and $y = 2$ we find $c_0 = 2$; from (*2*) with $x = 0$ and $y' = 1$ we find $c_1 = 1$. Since the third condition requires $y'' = -2y'$, we set

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \cdots = -2c_1 - 4c_2x - 6c_3x^2 - 8c_4x^3 - \cdots$$

from which it follows that $c_2 = -c_1 = -1$, $c_3 = -\frac{2}{3}c_2 = \frac{2}{3}$, $c_4 = -\frac{1}{2}c_3 = -\frac{1}{3}, \ldots$. Thus, $y = 2 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \cdots$ is the required series.

2. Assuming that $f(x)$ together with its derivatives of all orders exist at $x = a$ and that $f(x)$ can be represented as a power series in $(x - a)$, show that this series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + \cdots$$

Let the series be

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_{n-1}(x-a)^{n-1} + \cdots \tag{1}$$

Differentiating successively, we obtain

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1} + \cdots \tag{2}$$
$$f''(x) = 2c_2 + 6c_3(x-a) + 12c_4(x-a)^2 + 20c_5(x-a)^3 + \cdots + (n+1)nc_{n+1}(x-a)^{n-1} + \cdots \tag{3}$$

$$f'''(x) = 6c_3 + 24c_4(x-a) + 60c_5(x-a)^2 + \cdots + (n+2)(n+1)nc_{n+2}(x-a)^{n-1} + \cdots \quad (4)$$

. . . .

Setting $x = a$ in (1), (2), (3), . . . , we find in turn

$$c_0 = f(a), \quad c_1 = f'(a), \quad c_2 = \frac{1}{2!} f''(a), \quad \ldots, \quad c_{n-1} = \frac{1}{(n-1)!} f^{(n-1)}(a), \quad \ldots$$

When these replacements are made in (1), we have the required Taylor's series.

In Problems 3 to 8, obtain the expansion of the function in powers of x or $x - a$ as indicated, under the assumptions of this chapter, and determine the interval of convergence of the series.

3. e^{-2x}; powers of x

We have

$$\begin{array}{ll} f(x) = e^{-2x} & f(0) = 1 \\ f'(x) = -2e^{-2x} & f'(0) = -2 \\ f''(x) = 2^2 e^{-2x} & f''(0) = 2^2 \\ f'''(x) = -2^3 e^{-2x} & f'''(0) = -2^3 \\ \cdots\cdots & \cdots\cdots \end{array}$$

Then

$$e^{-2x} = 1 - 2x + \frac{2^2}{2!}x^2 - \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 - \cdots + (-1)^n \frac{2^n}{n!}x^n \cdots$$

and since

$$\lim_{n\to+\infty} \left| \frac{2^{n+1}x^{n+1}}{(n+1)!} \frac{n!}{2^n x^n} \right| = |x| \lim_{n\to+\infty} \frac{2}{n+1} = 0$$

the series converges for every value of x.

4. $\sin x$; powers of x

We have

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f'''(x) = -\cos x & f'''(0) = -1 \\ \cdots\cdots & \cdots\cdots \end{array}$$

The values of the derivatives at $x = 0$ form cycles of 0, 1, 0, -1; hence

$$\sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} \cdots$$

and since

$$\lim_{n\to+\infty} \left| \frac{x^{2n+1}}{(2n+1)!} \frac{(2n-1)!}{x^{2n-1}} \right| = x^2 \lim_{n\to+\infty} \frac{1}{2n(2n+1)} = 0$$

the series converges for every value of x.

5. $\ln(1+x)$; powers of x

Here

$$\begin{array}{ll} f(x) = \ln(1+x) & f(0) = 0 \\ f'(x) = \dfrac{1}{1+x} & f'(0) = 1 \\ f''(x) = -\dfrac{1}{(1+x)^2} & f''(0) = -1 \end{array}$$

$$f'''(x) = \frac{1\cdot 2}{(1+x)^3} \qquad f'''(0) = 2!$$

$$f^{iv}(x) = -\frac{1\cdot 2\cdot 3}{(1+x)^4} \qquad f^{iv}(0) = -3!$$

$$\cdots\cdots\cdots \qquad \cdots\cdots$$

Hence
$$\ln(1+x) = x - \frac{x^2}{2!} + 2!\,\frac{x^3}{3!} - 3!\,\frac{x^4}{4!} + \cdots + (-1)^{n-1}(n-1)!\,\frac{x^n}{n!}\cdots$$
$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n-1}\frac{1}{n}x^n\cdots$$

By Problem 1 of Chapter 57, the series converges on the interval $-1 < x \le 1$.

6. $\arctan x$; powers of x

We have

$$
\begin{aligned}
f(x) &= \arctan x & f(0) &= 0\\
f'(x) &= \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots & f'(0) &= 1\\
f''(x) &= -2x + 4x^3 - 6x^5 + \cdots & f''(0) &= 0\\
f'''(x) &= -2 + 12x^2 - 30x^4 + \cdots & f'''(0) &= -2!\\
f^{iv}(x) &= 24x - 120x^3 + \cdots & f^{iv}(0) &= 0\\
f^{v}(x) &= 24 - 360x^2 + \cdots & f^{v}(0) &= 4!\\
f^{vi}(x) &= -720x + \cdots & f^{vi}(0) &= 0\\
f^{vii}(x) &= -720 + \cdots & f^{vii}(0) &= -6!\\
&\cdots\cdots\cdots & &\cdots\cdots
\end{aligned}
$$

So
$$\arctan x = x - \frac{2!}{3!}x^3 + \frac{4!}{5!}x^5 - \frac{6!}{7!}x^7 + \cdots$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}\cdots$$

From Problem 7 of Chapter 57, the interval of convergence is $-1 \le x \le 1$.

7. $e^{x/2}$; powers of $x-2$

We have

$$
\begin{aligned}
f(x) &= e^{x/2} & f(2) &= e\\
f'(x) &= \tfrac{1}{2}e^{x/2} & f'(2) &= \tfrac{1}{2}e\\
f''(x) &= \tfrac{1}{4}e^{x/2} & f''(2) &= \tfrac{1}{4}e\\
&\cdots\cdots & &\cdots\cdots
\end{aligned}
$$

Hence
$$e^{x/2} = e\left[1 + \frac{1}{2}(x-2) + \frac{1}{4}\frac{(x-2)^2}{2!} + \cdots + \frac{1}{2^{n-1}}\frac{(x-2)^{n-1}}{(n-1)!} + \cdots\right]$$

and since
$$\lim_{n\to+\infty}\left|\frac{(x-2)^n}{2^n n!}\,\frac{2^{n-1}(n-1)!}{(x-2)^{n-1}}\right| = \frac{1}{2}|x-2|\lim_{n\to+\infty}\frac{1}{n} = 0$$

the series converges for every value of x.

8. $\ln x$; powers of $x-2$

Here

$$
\begin{aligned}
f(x) &= \ln x & f(2) &= \ln 2\\
f'(x) &= x^{-1} & f'(2) &= \tfrac{1}{2}\\
f''(x) &= -x^{-2} & f''(2) &= -\tfrac{1}{4}
\end{aligned}
$$

$$f'''(x) = 2x^{-3} \qquad f'''(2) = \tfrac{1}{4}$$
$$f^{iv}(x) = -6x^{-4} \qquad f^{iv}(2) = -\tfrac{3}{8}$$
$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

So
$$\ln x = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4}\frac{(x-2)^2}{2!} + \frac{1}{4}\frac{(x-2)^3}{3!} - \frac{3}{8}\frac{(x-2)^4}{4!} + \cdots$$
$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \cdots$$

Since
$$\lim_{n\to+\infty}\left|\frac{(x-2)^{n+1}}{2^{n+1}(n+1)}\,\frac{2^n n}{(x-2)^n}\right| = \frac{1}{2}|x-2|\lim_{n\to+\infty}\frac{n}{n+1} = \frac{1}{2}|x-2|$$

the series converges for $|x-2|<2$ or $0<x<4$.

For $x=0$, the series is $\ln 2 -$ (harmonic series) and diverges; for $x=4$, the series is $\ln 2 + 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots$ and converges. Thus the series converges on the interval $0<x\le 4$.

9. Obtain the Maclaurin's series expansion for $\sqrt{1+\sin x} = \sin\frac{1}{2}x + \cos\frac{1}{2}x$.

Replace x by $\frac{1}{2}x$ in the expansion for $\sin x$ (Problem 4) to obtain

$$\sin\frac{1}{2}x = \frac{1}{2}x - \frac{x^3}{2^3\cdot 3!} + \frac{x^5}{2^5\cdot 5!} - \frac{x^7}{2^7\cdot 7!} + \cdots$$

Differentiate this expansion to obtain

$$\cos\frac{1}{2}x = 2\left(\frac{1}{2} - \frac{x^2}{2^3\cdot 2!} + \frac{x^4}{2^5\cdot 4!} - \frac{x^6}{2^7\cdot 6!} + \cdots\right) = 1 - \frac{x^2}{2^2\cdot 2!} + \frac{x^4}{2^4\cdot 4!} - \frac{x^6}{2^6\cdot 6!} + \cdots$$

Then
$$\sqrt{1+\sin x} = \sin\frac{1}{2}x + \cos\frac{1}{2}x = 1 + \frac{x}{2} - \frac{x^2}{2^2\cdot 2!} - \frac{x^3}{2^3\cdot 3!} + \frac{x^4}{2^4\cdot 4!} + \frac{x^5}{2^5\cdot 5!} - \cdots$$

for all values of x.

10. Obtain the Maclaurin's series expansion for $e^{\cos x} = e(e^{(\cos x - 1)})$.

Using $e^u = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \cdots$ and $u = \cos x - 1 = -\dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$, we find

$$e^{\cos x} = e\left[1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \frac{1}{2!}\left(\frac{x^4}{(2!)^2} - \frac{2x^6}{2!4!} + \cdots\right) + \frac{1}{3!}\left(-\frac{x^6}{(2!)^3} + \cdots\right) + \cdots\right]$$
$$= e\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31}{720}x^6 + \cdots\right)$$

11. Under the assumption that all necessary operations are valid, show that (*a*) $e^{ix} = \cos x + i\sin x$, (*b*) $e^{-ix} = \cos x - i\sin x$, (*c*) $\sin x = (e^{ix} - e^{-ix})/2i$, (*d*) $\cos x = (e^{ix} + e^{-ix})/2$, where $i = \sqrt{-1}$.

Since $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \dfrac{z^4}{4!} + \dfrac{z^5}{5!} + \cdots$, we have the following:

(*a*) $$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}\cdots$$
$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \cos x + i\sin x$$

(*b*) $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$

(*c*) $e^{ix} - e^{-ix} = 2i\sin x$; hence, $\sin x = (e^{ix} - e^{-ix})/2i$

(*d*) $e^{ix} + e^{-ix} = 2\cos x$; hence, $\cos x = (e^{ix} + e^{-ix})/2$

Supplementary Problems

12. Verify that (*a*) series (*58.5*) and (*58.6*) converge for $|x|<1$; (*b*) series (*58.7*) converges for $-1 \le x < 1$; (*c*) series (*58.8*) converges for $-1 \le x \le 1$.

13. Verify that (*a*) the series obtained by adding (*58.5*) and (*58.6*) converges for $|x|<1$; (*b*) the series obtained by adding (*58.7*) and (*58.8*) converges for $-1 \le x < 1$.

14. Find the power series $y = \Sigma\, c_n x^n$ satisfying the conditions (1) $y = 2$ when $x = 0$, (2) $y' = 0$ when $x = 0$, and (3) $y'' - y = 0$. *Ans.* $y = 2 + x^2 + \dfrac{x^4}{12} + \cdots + \dfrac{2x^{2n}}{(2n)!} + \cdots$

15. Find the power series $y = \Sigma\, c_n x^n$ sastisfying the conditions (1) $y = 1$ when $x = 0$, (2) $y' = 1$ when $x = 0$, and (3) $y'' + y = 0$. *Ans.* $y = 1 + x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} - \cdots$

16. Obtain the given Maclaurin's series expansion:

(*a*) $\cos^2 x = 1 - \dfrac{2}{2!}x^2 + \dfrac{2^3}{4!}x^4 - \cdots + (-1)^n \dfrac{2^{2n-1}}{(2n)!}x^{2n} - \cdots$, for all x

(*b*) $\sec x = 1 + \dfrac{1}{2}x^2 + \dfrac{5}{24}x^4 + \dfrac{61}{720}x^6 + \cdots$, for $-\pi/2 < x < \pi/2$

(*c*) $\tan x = x + \dfrac{1}{3}x^3 + \dfrac{2}{15}x^5 + \dfrac{17}{315}x^7 + \cdots$, for $-\pi/2 < x < \pi/2$

(*d*) $\arcsin x = x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\dfrac{x^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\dfrac{x^7}{7} + \cdots$, for $-1 < x < 1$

(*e*) $\sin^2 x = \dfrac{2}{2!}x^2 - \dfrac{2^3}{4!}x^4 + \dfrac{2^5}{6!}x^6 - \cdots + (-1)^{n+1}\dfrac{2^{2n-1}}{(2n)!}x^{2n} + \cdots$, for all x

17. Obtain the given Taylor's series expansion:

(*a*) $e^x = e^a\left[1 + (x-a) + \dfrac{(x-a)^2}{2!} + \dfrac{(x-a)^3}{3!} + \cdots + \dfrac{(x-a)^{n-1}}{(n-1)!} + \cdots\right]$, for all x

(*b*) $\sin x = \sin a + (x-a)\cos a - \dfrac{(x-a)^2}{2!}\sin a - \dfrac{(x-a)^3}{3!}\cos a + \cdots$, for all x

(*c*) $\cos x = \dfrac{1}{\sqrt{2}}\left[1 - (x - \tfrac{1}{4}\pi) - \dfrac{(x-\frac{1}{4}\pi)^2}{2!} + \dfrac{(x-\frac{1}{4}\pi)^3}{3!} + \cdots\right]$, for all x

18. Differentiate the expansion for $\sin x$ (Problem 4) to obtain the expansion for $\cos x$. Then identify the solution of Problem 15 as $y = \sin x + \cos x$.

19. Replace x by $\frac{1}{2}x$ in the expansion for e^{-2x} (Problem 3) to obtain the expansion for e^{-x}. In this latter series replace x by $-x$ to obtain the expansion for e^x; then identify the solution of Problem 14 as $y = e^x + e^{-x}$.

20. Obtain the Maclaurin's series expansion $\sin^2 x = (\sin x)^2 = x^2 - \dfrac{2x^4}{3!} + \dfrac{32x^6}{3!5!} - \dfrac{96x^8}{3!7!} + \cdots$, for all x.

21. Show that $\displaystyle\int_0^x e^{-y^2}\,dy = x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots$, for all x.

22. Obtain by division the series expansion of $\dfrac{1}{1+x^2}$; then obtain

$$\arctan x = \int_0^x \frac{dx}{1+x^2} = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots$$

and compare with the result of Problem 6.

23. By the binomial theorem, establish $\dfrac{1}{\sqrt{1-x^2}} = 1 + \dfrac{1}{2}x^2 + \dfrac{1\cdot 3}{2\cdot 4}x^4 + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$; then obtain

$$\arcsin x = \int_0^x \frac{dx}{\sqrt{1-x^2}} = x + \frac{1\cdot x^3}{2\cdot 3} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5\cdot x^7}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

24. Multiply the respective series expansions to obtain (*a*) $e^x \sin x = x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30} - \dfrac{x^6}{90}\cdots$; (*b*) $e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} - \dfrac{x^5}{30}\cdots$.

25. Write $\sec x = \dfrac{1}{\cos x} = \dfrac{1}{1 - x^2/2! + x^4/4! - \cdots} = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$. Multiply the two series and equate to zero the coefficient of each positive power of x to obtain $c_0 = 1$, $c_1 = 0, \ldots$.

Chapter 59

Maclaurin's and Taylor's Formulas with Remainders

MACLAURIN'S FORMULA. If $f(x)$ and its first n derivatives are continuous on an interval containing $x = 0$, then there are numbers x_0 and x_0^* between 0 and x such that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + R_n(x)$$

where $$R_n(x) = \frac{f^{(n)}(x_0)}{n!}x^n \qquad \text{(Lagrange form)}$$

or $$R_n(x) = \frac{f^{(n)}(x_0^*)}{(n-1)!}(x - x_0^*)^{n-1}x \qquad \text{(Cauchy form)}$$

TAYLOR'S FORMULA. If $f(x)$ and its first n derivatives are continuous on an interval containing $x = a$, then there are numbers x_0 and x_0^* between a and x such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x)$$

where $$R_n(x) = \frac{f^{(n)}(x_0)}{n!}(x-a)^n \qquad \text{(Lagrange form)}$$

or $$R_n(x) = \frac{f^{(n)}(x_0^*)}{(n-1)!}(x - x_0^*)^{n-1}(x-a) \qquad \text{(Cauchy form)}$$

Maclaurin's formula is a special case ($a = 0$) of Taylor's formula. Taylor's formula with the Lagrange form of the remainder is a simple variation of the extended law of the mean (see Chapter 26). For the derivation of the formula with the Cauchy form of the remainder, see Problem 10.

The Maclaurin's and Taylor's series expansions of a function $f(x)$ as obtained in Chapter 58 represent that function for those values, and only those values, of x for which $\lim_{n\to+\infty} R_n(x) = 0$.

SERIES FOR REFERENCE. The following series, with the functions they represent and the intervals on which they do so, are listed here for reference:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \cdots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \quad \text{for all } x$$

$$\ln(a+x) = \ln a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \cdots + (-1)^{n-1}\frac{x^n}{na^n} + \cdots \quad \text{for } -a < x \le a$$

$$\arcsin x = x + \frac{1\cdot x^3}{2\cdot 3} + \frac{1\cdot 3\cdot x^5}{2\cdot 4\cdot 5} + \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-3)x^{2n-1}}{2\cdot 4\cdot 6\cdots(2n-2)(2n-1)} + \cdots \quad \text{for } -1 \le x \le 1$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \quad \text{for } -1 \le x \le 1$$

$$\ln x = \ln a + \frac{1}{a}(x-a) - \frac{1}{2a^2}(x-a)^2 + \frac{1}{3a^3}(x-a)^3 - \cdots + \frac{(-1)^n}{(n-1)a^{n-1}}(x-a)^{n-1} - \cdots \quad \text{for } 0 < x \le 2a$$

$$e^x = e^a\left[1 + (x-a) + \frac{(x-a)^2}{2!} + \frac{(x-a)^3}{3!} + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} + \cdots\right] \quad \text{for all } x$$

$$\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a + \cdots \quad \text{for all } x$$

$$\cos x = \cos a - (x-a)\sin a - \frac{(x-a)^2}{2!}\cos a + \frac{(x-a)^3}{3!}\sin a + \cdots \quad \text{for all } x$$

Solved Problems

1. Find the interval for which e^x may be represented by its Maclaurin's series.

$f^{(n)}(x) = e^x$; the Lagrange form of the remainder is $|R_n(x)| = \left|\frac{x^n}{n!} f^{(n)}(x_0)\right| = \frac{|x^n|}{n!} e^{x_0}$, where x_0 is between 0 and x.

The factor $\frac{x^n}{n!}$ is a general term of $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ which is known to converge for every value of x. Thus, $\lim_{n\to+\infty} \frac{|x^n|}{n!} = 0$. The factor e^{x_0} is bounded by the maximum of e^x and 1. Hence, $\lim_{n\to+\infty} R_n(x) = 0$ and the series represents e^x for all values of x.

2. Find the interval for which $\sin x$ may be represented by its Maclaurin's series.

Apart from sign, $f^{(n)}(x) = \sin x$ or $\cos x$, and $|R_n(x)| = \frac{|x^n|}{n!}|\sin x_0|$ or $\frac{|x^n|}{n!}|\cos x_0|$, where x_0 is between 0 and x.

As in Problem 1, $\frac{x^n}{n!} \to 0$ as $n \to +\infty$. Since $|\sin x_0|$ and $|\cos x_0|$ are never greater than 1, $\lim_{n\to+\infty} R_n(x) = 0$ and the series represents $\sin x$ for all values of x.

3. Find the interval for which $\cos x$ may be represented by its Taylor's series in powers of $(x - a)$.

For the Lagrange form of the remainder, we have $|R_n(x)| = \frac{|(x-a)^n|}{n!}|\sin x_0|$ or $\frac{|(x-a)^n|}{n!}|\cos x_0|$, where x_0 is between a and x.

Since $\frac{|(x-a)^n|}{n!} \to 0$ as $n \to +\infty$, while $|\sin x_0|$ and $|\cos x_0|$ are never greater than 1, $\lim_{n\to+\infty} R_n(x) = 0$ and the series represents $\cos x$ for all values of x.

4. Find the interval for which $\ln(1 + x)$ may be represented by its Maclaurin's series.

Here $f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$; then with x_0 and x_0^* between 0 and x, the Lagrange form of the remainder is

$$R_n(x) = (-1)^{n-1}\frac{x^n}{n!}\frac{(n-1)!}{(1+x_0)^n} = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+x_0}\right)^n \qquad (1)$$

and the Cauchy form of the remainder is

$$R_n(x) = (-1)^{n-1}\frac{(x-x_0^*)^{n-1}}{(n-1)!}\frac{(n-1)!}{(1+x_0^*)^n}x = (-1)^{n-1}\frac{x(x-x_0^*)^{n-1}}{(1+x_0^*)^n} \tag{2}$$

When $0<x_0<x\le 1$, then $0<x<1+x_0$ and $\dfrac{x}{1+x_0}<1$; then, from (*1*),

$$|R_n(x)| = \frac{1}{n}\left(\frac{x}{1+x_0}\right)^n < \frac{1}{n} \qquad \text{and} \qquad \lim_{n\to+\infty} R_n(x)=0$$

When $-1<x<x_0^*<0$, then $0<1+x<1+x_0^*$ and $\dfrac{1}{1+x_0^*}<\dfrac{1}{1+x}$. From (*2*),

$$|R_n(x)| = \frac{|x-x_0^*|^{n-1}}{(1+x_0^*)^n}|x| = \left|\frac{x_0^*-x}{1+x_0^*}\right|^{n-1}\frac{|x|}{1+x_0^*} = \left(\frac{x_0^*+|x|}{1+x_0^*}\right)^{n-1}\frac{|x|}{1+x_0^*} < \left(\frac{x_0^*+|x|}{1+x_0^*}\right)^{n-1}\frac{|x|}{1+x}$$

Now since $x_0^*<0$ and $1>|x|$, we have $x_0^*<x_0^*|x|$, $x_0^*+|x|<|x|+x_0^*|x|$, and $\dfrac{x_0^*+|x|}{1+x_0^*}<|x|$. Thus,

$$|R_n(x)| < \frac{|x|^n}{1+x} \qquad \text{and} \qquad \lim_{n\to+\infty} R_n(x)=0$$

Hence, $\ln(1+x)$ is represented by its Maclaurin's series on the interval $-1<x\le 1$.

5. For the Maclaurin's series representing e^x, show that

$$|R_n(x)| < \frac{|x^n|}{n!} \text{ when } x<0 \qquad \text{and} \qquad R_n(x) < \frac{x^n e^x}{n!} \text{ when } x>0$$

From Problem 1, $R_n(x) = \dfrac{x^n}{n!}e^{x_0}$, where x_0 is between 0 and x. When $x<0$, $e^{x_0}<1$; hence, $|R_n(x)|<\dfrac{|x^n|}{n!}$. When $x>0$, $e^{x_0}<e^x$; hence, $R_n(x)<\dfrac{x^n e^x}{n!}$.

6. For the Maclaurin's series representing $\ln(1+x)$, show that

$$R_n(x) < \frac{x^n}{n} \text{ when } 0<x\le 1 \qquad \text{and} \qquad |R_n(x)| < \frac{|x^n|}{n(1+x)^n} \text{ when } -1<x<0$$

From (*1*) of Problem 4, $|R_n(x)| = \dfrac{1}{n}\left|\dfrac{x}{1+x_0}\right|^n$, where x_0 is between 0 and x. When $0<x_0<x\le 1$, $\dfrac{1}{1+x_0}<1$; hence, $|R_n(x)<\dfrac{x^n}{n}$. When $-1<x<x_0<0$, $1+x_0>1+x$ and $\dfrac{1}{1+x_0}<\dfrac{1}{1+x}$; hence, $|R_n(x)|<\dfrac{|x^n|}{n(1+x)^n}$.

Supplementary Problems

7. Find the interval for which $\cos x$ may be represented by its Maclaurin's series.

Ans. all values of x

8. Find the intervals for which (*a*) e^x and (*b*) $\sin x$ may be represented by their Taylor's series in powers of $(x-a)$. *Ans.* all values of x

9. Show that $\ln x$ may be represented by its Taylor's series in powers of $(x-a)$ on the interval $0<x\le 2a$. $\left(\textit{Hint}: |R_n(x)| = \left|\dfrac{(x-a)(x-x_0^*)^{n-1}}{(x_0^*)^n}\right|. \text{ For } 0<x<a \text{ and for } a<x\le 2a, \left|\dfrac{x-x_0^*}{x_0^*}\right|<1.\right)$

10. Let T be defined by

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + T(b-a)$$

and define

$$F(x) = -f(b) + f(x) + \frac{f'(x)}{1!}(b-x) + \frac{f''(x)}{2!}(b-x)^2 + \cdots + \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{n-1} + T(b-x)$$

Carry through as in Problem 15 of Chapter 26, and obtain Taylor's formula with the Cauchy form of the remainder.

11. (*a*) In the Cauchy form of the remainder of Taylor's formula, put $x_0^* = a + \theta(x-a)$, where $0 < \theta < 1$. Show that $R_n(x) = \dfrac{f^{(n)}[a + \theta(x-a)]}{(n-1)!}(1-\theta)^{n-1}(x-a)^n$.

(*b*) Show that $R_n(x) = \dfrac{f^{(n)}(\theta x)}{(n-1)!}(1-\theta)^{n-1}x^n$ in Maclaurin's formula.

12. Show that $\dfrac{1}{1-x}$ is represented by its Maclaurin's series on the interval $-1 \le x < 1$. $\Big($*Hint*: From Problem 11(*b*), $R_n(x) = \dfrac{n(1-\theta)^{n-1}x^n}{(1-\theta x)^{n+1}}$ for $0 < \theta < 1$. For $|x| < 1$, $\dfrac{1-\theta}{1-\theta x} < 1$ and $1 - \theta x > 1 - |x|$.$\Big)$

13. (*a*) Show that $xe^x = \sum_{i=1}^{+\infty} \dfrac{n}{n!} x^n$ for all values of x, and $\sum_{i=1}^{+\infty} \dfrac{n}{n!} = e$; also show that $(x^2 + x)e^x = \sum_{i=1}^{+\infty} \dfrac{n^2}{n!} x^n$ and $\sum_{i=1}^{+\infty} \dfrac{n^2}{n!} = 2e$. (*b*) Obtain $\sum_{i=1}^{+\infty} \dfrac{n^3}{n!} = 5e$ and $\sum_{i=1}^{+\infty} \dfrac{n^4}{n!} = 15e$.

Chapter 60

Computations Using Power Series

TABLES OF LOGARITHMS, trigonometric functions, and such are computed by means of power series. Other uses of series are suggested in the problems below.

It is usually necessary to have some estimate of how well the sum of the first n terms of a series represents the corresponding function for a given value of the variable. For this purpose two theorems from preceding chapters are useful:

1. If $f(x)$ is represented by an alternating series, and if $x = \xi$ is on its interval of convergence, the error introduced by using the sum of the values of the first n terms as an approximate value of $f(\xi)$ does not exceed the numerical value of the first term discarded.
2. If $f(x)$ is represented by its Taylor's series, and if $x = \xi$ is on its interval of convergence, the error introduced by using the sum of the values of the first n terms as an approximate value of $f(\xi)$ does not exceed $|x - a|^n M/n!$, where M is equal to the maximum value of $|f^{(n)}(x)|$ on the interval a to ξ. For a Maclaurin's series, $a = 0$.

CORRECTNESS OF APPROXIMATIONS. If an actual value V is approximated by a number A, we say that the approximation is *correct to k decimal places* if the error $|A - V|$ is less than $5 \times 10^{k+1}$. This is equivalent to saying that A would be the result of rounding off V to k decimal places.

Solved Problems

1. Find the value of $1/e$ correct to five decimal places.

Since $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!} + \cdots$$

we have $$\begin{aligned} e^{-1} &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \\ &= 1 - 1 + 0.500\,000 - 0.166\,667 + 0.041\,667 - 0.008\,333 + 0.001\,389 \\ &\qquad - 0.000\,198 + 0.000\,025 - 0.000\,003 + \cdots \\ &= 0.36788 \end{aligned}$$

2. Find the value of $\sin 62°$ correct to five decimal places.

The Taylor's for $\sin x$ series in powers of $(x - a)$ is

$$\sin x = \sin a + (x - a)\cos a - \frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a + \cdots$$

Take $a = 60°$, since it is near $62°$ and its trigonometric functions are known. Then $x - a = 62° - 60° = 2° = \pi/90 = 0.034\,907$ and

$$\begin{aligned} \sin 62° &= \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}(0.034\,907) - \tfrac{1}{4}\sqrt{3}(0.034\,907)^2 - \tfrac{1}{12}(0.034\,907)^3 + \cdots \\ &= 0.866\,025 + 0.017\,454 - 0.000\,528 - 0.000\,004 + \cdots = 0.88295 \end{aligned}$$

3. Find the value of ln 0.97 correct to seven decimal places.

For $$\ln(a-x) = \ln a - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \cdots - \frac{x^n}{na^n} - \cdots$$
we take $a = 1$ and $x = 0.03$; then
$$\ln 0.97 = -0.03 - \tfrac{1}{2}(0.03)^2 - \tfrac{1}{3}(0.03)^3 - \tfrac{1}{4}(0.03)^4 - \tfrac{1}{5}(0.03)^5 - \cdots = -0.030\,459\,2$$

4. How many terms in the expansion of $\ln(1+x)$ must be used to ensure finding ln 1.02 with an error not exceeding 0.000 000 05?

We have $$\ln 1.02 = 0.02 - \frac{(0.02)^2}{2} + \frac{(0.02)^3}{3} - \frac{(0.02)^4}{4} + \cdots$$
Since this is an alternating series, the error introduced by discarding all terms after the first n is not greater than the numerical value of the first term discarded. The problem here is to find the first term whose numerical value is less than 0.000 000 05. This must be done by trial. Since $(0.02)^3/3 = 0.000\,002\,7$ and $(0.02)^4/4 = 0.000\,000\,04$, the desired accuracy is obtained when the first three terms are used.

5. For what values of x can $\sin x$ be replaced by x, if the allowable error is 0.0005?

$\sin x = x - x^3/3! + x^5/5! - \cdots$ is an alternating series. The error in using only the first term x is thus less than $|x^3|/3!$. Now $|x^3|/3! = 0.0005$ requires $|x^3| = 0.003$ or $|x| = 0.1442$; thus, $|x| < 8°15'$.

6. How large may the angle be taken if the values of $\cos x$ are to be computed using three terms of the Taylor's series in powers of $(x - \pi/3)$ and the error must not exceed 0.00005?

Since $f'''(x) = \sin x$, $|R_3| = \dfrac{|\sin x_0|}{3!}|x - \pi/3|^3$, where x_0 is between $\pi/3$ and x.
Since $|\sin x_0| \le 1$, $|R_3| \le \frac{1}{6}|x - \pi/3|^3 = 0.00005$.
Then $|x - \pi/3| \le \sqrt[3]{0.0003} = 0.0669 = 3°50'$. Thus x may have any value between 56°10′ and 63°50′.

7. Approximate the amount by which an arc of a great circle on the earth 100 miles long will recede from its chord.

Let x be the required amount. From Fig. 60-1, $x = OB - OA = R - R\cos\alpha$, where R is the radius of the earth. Since angle α is small, $\cos\alpha = 1 - \frac{1}{2}x^2$, approximately, and
$$x = R\left[1 - \left(1 - \frac{1}{2}\alpha^2\right)\right] = \frac{1}{2}R\alpha^2 = \frac{(R\alpha)^2}{2R} = \frac{(50)^2}{2R}$$
Taking $R = 4000$ mi yields $x = \frac{5}{16}$ mi.

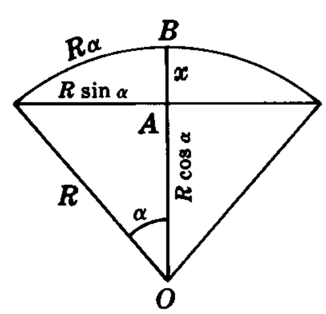

Fig. 60-1

8. Derive the approximation formula $\sin(\frac{1}{4}\pi + x) = \frac{1}{2}\sqrt{2}(1+x)$, and use it to find sin 43°.

Using the first two terms of the Taylor's expansion, we have

$$\sin\left(\tfrac{1}{4}\pi + x\right) = \sin\tfrac{1}{4}\pi + x\cos\tfrac{1}{4}\pi = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}x = \tfrac{1}{2}\sqrt{2}(1+x)$$
$$\sin 43^\circ = \sin\left[\tfrac{1}{4}\pi + (-\pi/90)\right] = \tfrac{1}{2}\sqrt{2}(1 - 0.0349) = 0.6824$$

9. Solve the equation $\cos x - 2x^2 = 0$.

Replace $\cos x$ by the first two terms, $1 - \frac{1}{2}x^2$, of its Maclaurin's series. Then the equation is

$$1 - \tfrac{1}{2}x^2 - 2x^2 = 0 \qquad \text{or} \qquad 2 - 5x^2 = 0$$

The roots are $\pm\sqrt{10}/5 = \pm 0.632$. Newton's method gives the roots as ± 0.635.

10. Use power series expansions to evaluate $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$.

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{\sin x} = \lim_{x\to 0}\frac{\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots\right) - \left(1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots\right)}{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}$$
$$= \lim_{x\to 0}\frac{2x + 2x^3/3! + \cdots}{x - x^3/3! + \cdots} = \lim_{x\to 0}\frac{2 + x^2/3 + \cdots}{1 - x^2/6 + \cdots} = 2$$

11. Expand $f(x) = x^4 - 11x^3 + 43x^2 - 60x + 14$ in powers of $(x-3)$, and find $\int_3^{3.2} f(x)\,dx$.

$f(3) = 5$, $f'(3) = 9$, $f''(3) = -4$, $f'''(3) = 6$, and $f^{\text{iv}}(3) = 24$. Hence,

$$f(x) = 5 + 9(x-3) - 2(x-3)^2 + (x-3)^3 + (x-3)^4$$

and
$$\int_3^{3.2} f(x)\,dx = \left[5x + \tfrac{9}{2}(x-3)^2 - \tfrac{2}{3}(x-3)^3 + \tfrac{1}{4}(x-3)^4 + \tfrac{1}{5}(x-3)^5\right]_3^{3.2} = 1.185$$

12. Evaluate $\int_0^1 \dfrac{\sin x}{x}\,dx$.

The difficulty here is that $\int \dfrac{\sin x}{x}\,dx$ cannot be expressed in terms of elementary functions. However,

$$\int_0^1 \frac{\sin x}{x}\,dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)dx = \int_0^1\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)dx$$
$$= \left[x - \frac{x^3}{3\cdot 3!} + \frac{x^5}{5\cdot 5!} - \frac{x^7}{7\cdot 7!} + \cdots\right]_0^1 = 0.946\,083$$

The error in using only four terms is $\le \dfrac{1}{9\cdot 9!} = 0.000\,000\,3$.

Supplementary Problems

13. Compute to four decimal places (*a*) e^{-2}; (*b*) $\sin 32^\circ$; (*c*) $\cos 36^\circ$; (*d*) $\tan 31^\circ$.

Ans. (*a*) 0.1353; (*b*) 0.5299; (*c*) 0.8090; (*d*) 0.6009

14. For what range of x can
(*a*) e^x be replaced by $1 + x + \frac{1}{2}x^2$ if the allowable error is 0.0005?
(*b*) $\cos x$ be replaced by $1 - \frac{1}{2}x^2$ if the allowable error is 0.0005?

(*c*) $\sin x$ be replaced by $x - x^3/6 + x^5/120$ if the allowable error is 0.00005?

Ans. (*a*) $|x| < 0.1$; (*b*) $|x| < 18°57'$; (*c*) $|x| < 47°$

15. Use power series expansions to evaluate (*a*) $\lim_{x \to 0} \frac{e - e^{\cos x}}{x^2}$; (*b*) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x^3}$; (*c*) $\lim_{x \to 0} \frac{\cosh x - \cos x}{\sinh x - \sin x}$.

Ans. (*a*) $e/2$; (*b*) $\frac{1}{6}$; (*c*) ∞

16. Evaluate (*a*) $\int_0^{\pi/2} (1 - \frac{1}{2}\sin^2\phi)^{-1/2}\, d\phi$; (*b*) $\int_0^1 \cos\sqrt{x}\, dx$; (*c*) $\int_0^{0.5} \frac{dx}{1 + x^4}$.

Ans. (*a*) 1.854; (*b*) 0.76355; (*c*) 0.4940

17. Find the length of the curve $y = x^3/3$ from $x = 0$ to $x = 0.5$. *Ans.* 0.5031

18. Find the area under the curve $y = \sin x^2$ from $x = 0$ to $x = 1$. *Ans.* 0.3103

Chapter 61

Approximate Integration

AN APPROXIMATE VALUE of $\int_a^b f(x)\,dx$ may be obtained by means of certain formulas and by the use of modern computers. Approximation procedures are necessary when ordinary integration is difficult, when the indefinite integral cannot be expressed in terms of elementary functions, or when the integrand $f(x)$ is defined by a table of values.

In Chapter 39, an approximation of $\int_a^b f(x)\,dx$ was obtained as the sum $S_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x$. In obtaining S_n we interpreted the definite integral as an area, divided the area into n strips, approximated the area of each strip as that of a rectangle, and summed the several approximations. The formulas developed below vary only as to the manner of approximating the areas of the strips.

TRAPEZOIDAL RULE. Let the area bounded above by the curve $y = f(x)$, below by the x axis, and laterally by the lines $x = a$ and $x = b$ be divided into n vertical strips, each of width $h = (b - a)/n$, as in Fig. 61-1. Consider the ith strip, bounded above by the arc $P_{i-1}P_i$ of $y = f(x)$. As an approximation of the area of this strip, we take

$$\tfrac{1}{2}h[f(a + (i - 1)h) + f(a + ih)]$$

the area of the trapezoid obtained by replacing the arc $P_{i-1}P_i$ by the straight line segment $P_{i-1}P_i$. When each strip is so approximated, we have (where $\approx$ is to be read "is approximately")

$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + f(a + h)] + \frac{h}{2}[f(a + h) + f(a + 2h)] + \cdots + \frac{h}{2}[f(a + (n - 1)h) + f(b)]$$

or
$$\int_a^b f(x)\,dx \approx \frac{h}{2}[f(a) + 2f(a + h) + 2f(a + 2h) + \cdots + 2f(a + (n - 1)h) + f(b)] \qquad (61.1)$$

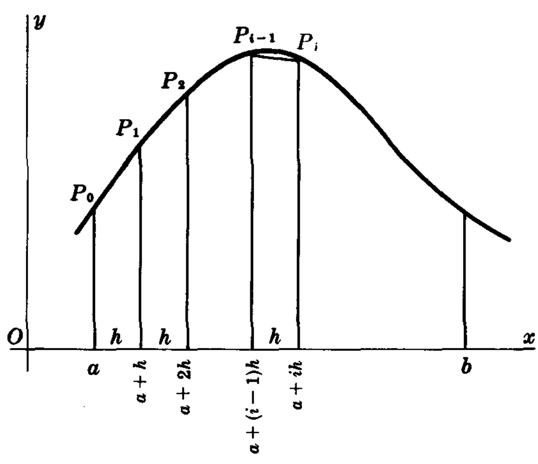

Fig. 61-1

PRISMOIDAL FORMULA. Let the area defined by $\int_a^b f(x)\,dx$ be separated into two vertical strips of width $h = \frac{1}{2}(b-a)$, and let the arc $P_0P_1P_2$ of $y = f(x)$ be replaced by the arc of the parabola $y = Ax^2 + Bx + C$ through the points P_0, P_1, P_2, as in Fig. 61-2. Then

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \qquad (61.2)$$

(See Problem 1.)

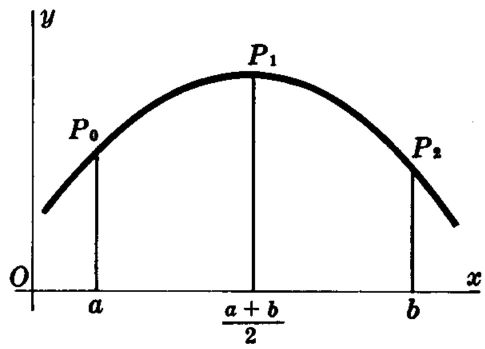

Fig. 61-2

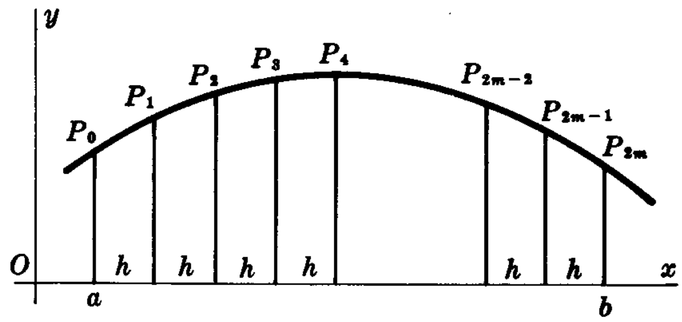

Fig. 61-3

SIMPSON'S RULE. Let the area under discussion be separated into $n = 2m$ strips, each of width $h = (b-a)/n$, as in Fig. 61-3. Using the prismoidal formula to approximate the area under each of the arcs $P_0P_1P_2$, $P_2P_3P_4, \ldots, P_{2m-2}P_{2m-1}P_{2m}$, we have

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\,[f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + 2f(a+4h)$$
$$+ \cdots + 2f(a+(2m-2)h) + 4f(a+(2m-1)h) + f(b)] \qquad (61.3)$$

POWER SERIES EXPANSION. This procedure for approximating $\int_a^b f(x)\,dx$ consists in replacing the integrand $f(x)$ by the first n terms of its Maclaurin's or Taylor's series. This method is available provided the integrand may be so expanded and the limits of integration fall within the interval of convergence of the series. (See Chapter 60.)

Solved Problems

1. For the parabola $y = Ax^2 + Bx + C$, passing through the points $P_0(\xi, y_0)$, $P_1\left(\frac{\xi+\eta}{2}, y_1\right)$, and $P_2(\eta, y_2)$ as shown in Fig. 61-4, show that $\int_\xi^\eta y\,dx = \frac{\eta-\xi}{6}(y_0 + 4y_1 + y_2)$.

We have $\int_\xi^\eta y\,dx = \int_\xi^\eta (Ax^2 + Bx + C)\,dx = \frac{A}{3}(\eta^3 - \xi^3) + \frac{B}{2}(\eta^2 - \xi^3) + c(\eta - \xi)$

$$= \frac{\eta-\xi}{3}\,[A(\xi^2 + \xi\eta + \eta^2) + \tfrac{3}{2}B(\xi+\eta) + 3C]$$

Now if $y = Ax^2 + Bx + C$ passes through the points P_0, P_1, P_2, then

$$y_0 = A\xi^2 + B\xi + C$$

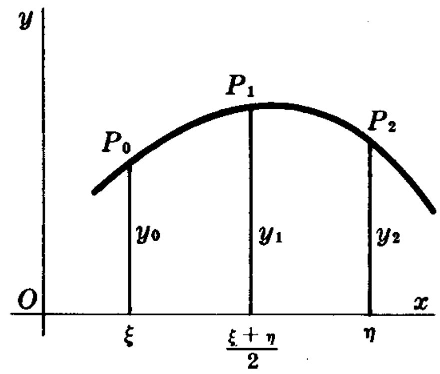

Fig. 61-4

$$y_1 = A\left(\frac{\xi+\eta}{2}\right)^2 + B\left(\frac{\xi+\eta}{2}\right) + C$$

$$y_2 = A\eta^2 + B\eta + C$$

and $$y_0 + 4y_1 + y_2 = 2[A(\xi^2 + \xi\eta + \eta^2) + \tfrac{3}{2}B(\xi+\eta) + 3C]$$

Thus, $\int_\xi^\eta y\,dx = \frac{\eta-\xi}{6}(y_0 + 4y_1 + y_2)$ as required.

2. Approximate $\int_0^{1/2} \frac{dx}{1+x^2}$ by each of the four methods, and check by integration.

Trapezoidal rule with $n = 5$: Here, $h = \frac{1/2 - 0}{5} = 0.1$. Then $a = 0$, $a + h = 0.1$, $a + 2h = 0.2$, $a + 3h = 0.3$, $a + 4h = 0.4$, and $b = 0.5$. Hence,

$$\int_0^{1/2} \frac{dx}{1+x^2} \approx \frac{0.1}{2}[f(0) + 2f(0.1) + 2f(0.2) + 2f(0.3) + 2f(0.4) + f(0.5)]$$

$$\approx \frac{1}{20}\left(1 + \frac{2}{1.01} + \frac{2}{1.04} + \frac{2}{1.09} + \frac{2}{1.16} + \frac{1}{1.25}\right) = 0.4631$$

Prismoidal formula: Here, $h = \frac{1/2-0}{2} = \frac{1}{4}$ and $f(a) = f(0) = 1$, $f\left(\frac{a+b}{2}\right) = f\left(\frac{1}{4}\right) = \frac{16}{17}$, and $f(b) = f\left(\frac{1}{2}\right) = \frac{4}{5}$. Then

$$\int_0^{1/2} \frac{dx}{1+x^2} \approx \frac{1}{3}\frac{1}{4}\left(1 + \frac{64}{17} + \frac{4}{5}\right) = \frac{1}{12}(1 + 3.76471 + 0.8) = 0.4637$$

Simpson's rule with $n = 4$: Here, $h = \frac{1/2-0}{4} = \frac{1}{8}$. Then $a = 0$, $a + h = \frac{1}{8}$, $a + 2h = \frac{1}{4}$, $a + 3h = \frac{3}{8}$, and $b = \frac{1}{2}$. Hence,

$$\int_0^{1/2} \frac{dx}{1+x} \approx \frac{1}{24}\left(1 + 4\frac{1}{1+(\frac{1}{8})^2} + 2\frac{1}{1+(\frac{1}{4})^2} + 4\frac{1}{1+(\frac{3}{8})^2} + \frac{1}{1+(\frac{1}{2})^2}\right)$$

$$\approx \frac{1}{24}\left(1 + \frac{256}{65} + \frac{32}{17} + \frac{256}{73} + \frac{4}{5}\right) = 0.4637$$

Series expansion, using seven terms

$$\int_0^{1/2} \frac{dx}{1+x^2} \approx \int_0^{1/2} (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12})\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{x^{13}}{13}\right]_0^{1/2}$$

$$\approx \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \frac{1}{9\cdot 2^9} - \frac{1}{11\cdot 2^{11}} + \frac{1}{13\cdot 2^{13}}$$

$$\approx 0.50000 - 0.04167 + 0.00625 - 0.00112 + 0.00022 - 0.00004 + 0.00001 = 0.4636$$

Integration: $$\int_0^{1/2} \frac{dx}{1+x^2} = [\arctan x]_0^{1/2} = \arctan \tfrac{1}{2} = 0.4636$$

3. Find the area bounded by $y = e^{-x^2}$, the x axis, and the lines $x = 0$ and $x = 1$ using (*a*) Simpson's rule with $n = 4$ and (*b*) series expansion.

(*a*) Here, $h = \frac{1}{4}$; since $a = 0$, $a + h = \frac{1}{4}$, $a + 2h = \frac{1}{2}$, $a + 3h = \frac{3}{4}$, and $b = 1$. Then

$$\int_0^1 e^{-x^2}\,dx \approx \frac{1/4}{3}(1 + 4e^{-1/16} + 2e^{-1/4} + 4e^{-9/16} + e^{-1})$$

$$\approx \tfrac{1}{12}[1 + 4(0.9399) + 2(0.7788) + 4(0.5701) + 0.3679] = 0.747 \text{ square units}$$

(*b*)
$$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!}\right) dx$$

$$\approx \left[x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!} + \frac{x^{13}}{13\cdot 6!}\right]_0^1$$

$$\approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \frac{1}{13\cdot 6!}$$

$$\approx 1 - 0.3333 + 0.1 - 0.0238 + 0.0046 - 0.0008 + 0.0001 = 0.747 \text{ square units}$$

4. A plot of land lies between a straight fence and a stream. At distances x from one end of the fence, the width of the plot y was measured (in yards) as follows:

x	0	20	40	60	80	100	120
y	0	22	41	53	38	17	0

Use Simpson's rule to approximate the area of the plot.

Here, $h = 20$ and

$$\int_0^{120} f(x)\,dx \approx \tfrac{20}{3}(0 + 4\cdot 22 + 2\cdot 41 + 4\cdot 53 + 2\cdot 38 + 4\cdot 17 + 0) \approx 3507 \text{ yd}^2$$

5. A certain curve is given by the following pairs of rectangular coordinates:

x	1	2	3	4	5	6	7	8	9
y	0	0.6	0.9	1.2	1.4	1.5	1.7	1.8	2

(*a*) Approximate the area between the curve, the x axis, and the lines $x = 1$ and $x = 9$, using Simpson's rule.
(*b*) Approximate the volume generated by revolving the area in (*a*) about the x axis, using Simpson's rule.

(*a*) Here, $h = 1$ and

$$\int_1^9 y\,dx \approx \tfrac{1}{3}[0 + 4(0.6) + 2(0.9) + 4(1.2) + 2(1.4) + 4(1.5) + 2(1.7) + 4(1.8) + 2]$$

$$\approx 10.13 \text{ square units}$$

(*b*)
$$\pi\int_1^9 y^2\,dx \approx \frac{\pi}{3}[0 + 4(0.6)^2 + 2(0.9)^2 + 4(1.2)^2 + 2(1.4)^2 + 4(1.5)^2 + 2(1.7)^2 + 4(1.8)^2 + 4]$$

$$\approx 46.58 \text{ cubic units}$$

Supplementary Problems

6. Derive Simpson's rule.

7. Approximate $\int_2^6 \frac{dx}{x}$ using (*a*) the trapezoidal rule with $n = 4$, (*b*) the prismoidal formula, and (*c*) Simpson's rule with $n = 4$. (*d*) Check by integration.

Ans. (*a*) 1.117; (*b*) 1.111; (*c*) 1.100; (*d*) 1.099

8. Approximate $\int_1^5 \sqrt{35 + x}\, dx$ as in Problem 7. *Ans.* (*a*) 24.654; (*b*) 24.655; (*c*) 24.655; (*d*) 24.655

9. Approximate $\int_1^3 \ln x\, dx$ using (*a*) the trapezoidal rule with $n = 5$ and (*b*) Simpson's rule with $n = 8$. (*c*) Check by integration. *Ans.* (*a*) 1.2870; (*b*) 1.2958; (*c*) 1.2958

10. Approximate $\int_0^1 \sqrt{1 + x^3}\, dx$ using (*a*) the trapezoidal rule with $n = 5$ and (*b*) Simpson's rule with $n = 4$. *Ans.* (*a*) 1.115; (*b*) 1.111

11. Approximate $\int_0^\pi \frac{\sin x}{x}\, dx$ by Simpson's rule with $n = 6$. *Ans.* 1.852

12. Use Simpson's rule to find (*a*) the area under the curve determined by the data below and (*b*) the volume generated by revolving the area about the x axis

x	1	2	3	4	5
y	1.8	4.2	7.8	9.2	12.3

Ans. (*a*) 27.8; (*b*) 228.44π